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Comparative Analysis of Euler and Order Four Runge-Kutta Methods in Adams-Bashforth-Moulton Predictor-Corrector Method

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ABSTRACT: This study conducts a comparative analysis of the Euler and Runge-Kutta 4 methods within the Adams Predictor-Corrector framework for solving second-order ordinary differential equations (ODEs) and coupled differential equations. The Euler method, a basic first-order explicit scheme, and the Runge-Kutta 4 method, a higher-order explicit scheme, are widely utilized numerical methods for ODEs. However, their effectiveness varies based on the problem's characteristics. We specifically investigate these methods integrated into the Adams Predictor-Corrector method, renowned for its stability and efficiency in solving initial value problems. Through both numerical experiments and theoretical analysis, we establish that the Runge-Kutta method demonstrates superior performance over the Euler method within this framework. Additionally, we observe that the 4th order of the Adams Predictor-Corrector method yields more accurate results than the 5th order for second-order ODEs, while the 5th order performs better for coupled ODEs. Overall, the Runge-Kutta method exhibits enhanced accuracy and stability compared to the Euler method in the context of the Adams Predictor-Corrector method

Keywords: Euler method, Runge-Kutta 4 method, Adams Predictor-Corrector method, Second-order ordinary differential equations (ODEs) and coupled differential equations.

1. INTRODUCTION

Numerical analysis involves approximating solutions to real-world problems rooted in algebra, geometry, and calculus. It serves as a crucial tool for scientists and engineers tackling complex issues that defy straightforward analytical solutions. Linear multistep methods, like the Adams-Bashforth and Adams-Moulton methods, enhance efficiency by leveraging and retaining information from preceding computational steps. These methods find extensive application across diverse fields such as engineering, physical sciences, life sciences, medicine, business, and the arts. [1,2,3]

Linear multistep methods optimize efficiency by preserving and utilizing data from previous computational steps rather than discarding it. This approach involves referencing several preceding data points and their derivatives. Specifically, linear multistep methods employ linear combinations of these historical data points and derivatives to approximate solutions, especially for initial value problems in ordinary differential equations (ODEs). In contrast, single-step methods rely solely on the immediate past data point and its derivatives to compute the current value.

There are several existing methods designed for solving this initial value problem:

$$y' = \frac{dy}{dx} = f(x, y); \qquad y(x_0) = y_0 \tag{1}$$

These methods include the Runge-Kutta and Euler methods, the Taylor series method as discussed by Lambert [4] .[5]Researched on Euler's method and the fourth-order Runge-Kutta Method (RK4) for solving initial value problems (IVPs) in ordinary differential equations (ODEs). [6] Study on starting the scheme using the fourth-order Runge-Kutta method and applying it to solve IVPs.[7] Assessment of the Adams-Bashforth-Moulton and Milne-Simpson methods for solving second-order ODEs with IVPs. [8] Use of the Adams-Moulton predictor-corrector method for solving ODEs with IVPs, comparing its stability and robustness to the Milne-Simpson method.

Linear multistep methods play a pivotal role in approximating solutions to differential equations, particularly when exact analytical solutions are impractical or unavailable [9]. By capitalizing on information from

multiple previous approximations, these methods enhance computational efficiency. Notable examples of linear multistep methods include the Adams-Bashforth, Adams-Moulton, Nytrom, and Milne-Simpson methods.

2. METHOD

This section deals with the description of the methods.

2.1. Description of Euler's Method

Let us consider the initial value problem

$$y' = \frac{dy}{dx} = f(x, y); \qquad y(x_0) = y_0 \tag{2}$$

The Euler formula is given by

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0,1,2,3,\cdots \tag{3}$$

2.2. The Description of the Fourth Order Runge Kutta method

Let us take the first order differential equation

$$y' = \frac{dy}{dx} = f(x, y); \qquad y(x_0) = y_0 \tag{4}$$

The general fourth-order Runge Kutta formulae interval is given by the followings:

$$\begin{aligned} k_1 &= hf(x_n, y_n) = df, \\ k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\ k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\ k_4 &= hf(x_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \tag{5}$$

2.3. Description of Adams Bashforth Methods

Consider the initial value problem

$$y'(x) = f(x, y), \tag{6}$$

Integrating (6) between the interval x_i and x_{i+1}, gives

$$\int_{x_i}^{x_{i+1}} y' \, dx = \int_{x_i}^{x_{i+1}} f(x, y) \, dx \tag{7}$$

Equation (7) is equivalently written as

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y) dx \tag{8}$$

The following are the explicit method through the k data values

$$(x_i, f_i), (x_{i+1}, f_{i+1}), (x_{i+2}, f_{i+2}), \cdots (x_{i-k+1}, f_{i-k+1}).$$

Then using Newton backward difference interpolating polynomial of degree $k-1$ and setting $x - x_i = hs$ gives

$$p_{k-1}(x) = f_i + s\nabla f_i + \frac{1}{2!}s(s+1)\nabla^2 f_i + \frac{1}{6}s(s+1)(s+2)\nabla^3 f_i + \cdots \frac{s(s+1)(s+2)\cdots(s+k-2)}{(k-1)!}\nabla^{k-1} f_i \tag{9}$$

noting that

$$s = \left[\frac{x - x_i}{h}\right] < 0$$

replacing $f(x, y)$ by p_{k-1} in the equation (9) yields

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} \left[f_i + s\nabla f_i + \frac{1}{2!}s(s+1)\nabla^2 f_i + \frac{1}{6}s(s+1)(s+2)\nabla^3 f_i + \cdots\right] dx \quad (10)$$

from

$$\begin{aligned} x - x_i &= h_s, \\ x &= x_i + h_s \\ \frac{dx}{ds} &= h \\ dx &= h\, ds \end{aligned}$$

When $s = 0, x = x_i$, when $s = 1, x = x_i + h$
We have:

$$y(x_{i+1}) = y(x_i) + \int_0^1 \left[f_i + s\nabla f_i + \frac{1}{2!}s(s+1)\nabla^2 f_i + \frac{1}{6}s(s+1)(s+2)\nabla^3 f_i + \cdots\right] ds \quad (11)$$

Integrating equation (11) with respect to s and neglecting higher derivatives gives

$$y_{i+1} = y_i + h\left(f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\nabla^2 f_i + \frac{3}{8}\nabla^3 f_i\right) \quad (12)$$

Applying the backward difference formula into equation (12) gives

$$y_{i+1} = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}) \quad (13)$$

The error term is then given as

$$TE = y(x_i + h) - y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}) = \frac{251}{720}h^5 y^5(\chi_4) \quad (14)$$

The above equation (13) is called the Adams Bashforth method of fourth order.

Adams Bashforth Method of Order 5
Extending equation (12) by one term and substitute the backward difference formula
We have:

$$y_{i+1} = y_i + \frac{h}{720}(1901f_i - 2774f_{i-1} + 2616f_{i-2} - 1274f_{i-3} + 251f_{i-4}) \quad (15)$$

The above equation (15) is called the Adams Bashforth Method of fifth order.

2.4 Description of Adams Moulton Methods

Consider the initial value problem

$$y'(x) = f(x, y), \quad (16)$$

Integrating (16) between the interval x_i and x_{i+1}, gives

$$\int_{x_i}^{x_{i+1}} y'\, dx = \int_{x_i}^{x_{i+1}} f\,(x, y)\, dx \quad (17)$$

Equation (17) is equivalently written as

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f\,(x, y) dx \quad (18)$$

The following are the implicit methods through the k data values

$$(x_{i+2}, f_{i+2}), (x_{i+1}, f_{i+1}), (x_i, f_i), \cdots, (x_{i-k+1}, f_{i-k+1})$$

Then using Newton backward difference interpolating polynomial of degree k and setting $hs = x - x_i$ gives

$$p_k(x) = f_{i+1} + (s-1)\nabla f_{i+1} + \frac{1}{2}s(s-1)\nabla^2 f_{i+1} + \frac{1}{6}s(s-1)(s+1)\nabla^3 f_{i+1} + \cdots + \frac{s(s-1)(s+1)\cdots(s+k-2)}{(k)!}\nabla^k f_{i+1} \quad (19)$$

nothing that

$$s = \left(\frac{x - x_i}{h}\right) < 0$$

Replacing $f(x, y)$ by p_k in equation (19) yield

$$y_{i+1} = y_i + h\int_{x_i}^{x_{i+1}} \left[f_{i+1} + (s-1)\nabla f_{i+1} + \frac{1}{2}(s-1)s\nabla^2 f_{i+1} + \frac{1}{6}(s-1)s(s+1)\nabla^3 f_i + \frac{1}{24}(s-1)s(s+1)(s+2)\nabla^4 f_{i+1} + \cdots\right] ds \quad (20)$$

from

$$\begin{aligned} x - x_i &= h_s \\ x &= x_i + h_s \\ \frac{dx}{ds} &= h \\ dx &= h\, ds \end{aligned}$$

When $s = -1, x = x_i - h$, when $s = 0, x = x_i$
We have:

$$y_{i+1} = y_i + h\int_{-1}^{0} \left[f_{i+1} + (s-1)\nabla f_{i+1} - \frac{1}{2}(s-1)s\nabla^2 f_{i+1} - \frac{1}{6}(s-1)s(s+1)\nabla^3 f_i + \cdots\right] ds \quad (21)$$

Integrating equation (21) with respect to s and neglecting higher derivatives gives

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\nabla^2 f_{i+1} - \frac{1}{24}\nabla^3 f_{i+1}\right] \quad (22)$$

Applying the backward difference formula into equation (22) gives

$$y_{i+1} = y_i + \frac{h}{24}(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}) \quad (23)$$

the error term is then given as

$$TE = y(x_i + h) - y_i + \frac{h}{24}(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}) = \frac{19}{720}h^5 y^5(\chi_4). \quad (24)$$

The above equation (23) is called the Adams-Moulton method of fourth order.

Adams Moulton Method of Order 5
Extending equation (22) by one term and substitute the backward difference formula
We have:

$$y_{i+1} = y_i + \frac{h}{720}[251f_{i+1} + 646f_i - 264f_{i-1} + 106f_{i-2} - 19f_{i-3}] \quad (25)$$

The above equation (25) is called the Adams-Moulton Method of fifth order.

3. RESULTS AND DISCUSSION

3.1. Application to second order differential equations

Problem 1 Statement
Solve the differential equation:

$$xy'' - 2y' = 10x^4 \tag{26}$$

with initial conditions $y(1) = 2$ and $y'(1) = 2$ over the interval $0 \leq x \leq 2$ with step sizes $h = 0.1$ and $h = 0.05$. Convert the second-order ODE into a system of first-order ODEs:

$$\begin{cases} y' = z \\ z' = \dfrac{10x^4 + 2z}{x} \end{cases}$$

with initial conditions $y(1) = 2$ and $z(2) = 2$.
The exact solutions are:

$$\begin{aligned} y &= x^5 - x^3 + 2, \\ z &= 5x^3 - 3x^2. \end{aligned} \tag{27}$$

Problem 2 Statement
Solve the differential equation:

$$y'' - 2y' = 4x \tag{28}$$

with initial conditions $y(0) = 1$ and $y'(0) = 2$ over the interval $0 \leq x \leq 2$ with step size $h = 0.1$. Convert the second-order ODE into a system of first-order ODEs: Let $y' = z$ and $y'' = z'$. The system becomes:

$$\begin{cases} y' = z \\ z' = 2y + 4x \end{cases}$$

Exact Solutions

The exact solutions to the differential equations are:

$$\begin{aligned} y &= -(x^2 + x) + 1.5e^{2x} - 0.5 \\ z &= -(2x + 1) + 3e^{2x} \end{aligned} \tag{29}$$

3.2. Application to Coupled differential equations

Problem 3 Statement
Solve the coupled differential equations:

$$\begin{cases} x' = x + 3y \\ y' = x - y \end{cases}$$

with initial conditions $x(0) = 2$ and $y(2) = 2$.
The exact solutions are:

$$\begin{aligned} x &= 3e^{2t} - e^{-2t}, \\ y &= e^{2t} + e^{-2t}. \end{aligned} \tag{30}$$

Problem 4 Statement
Solve the coupled differential equations:

$$\begin{cases} x' = 5x - 3y \\ y' = -6x + 2y \end{cases}$$

with initial conditions $x(1) = 1$ and $y(1) = 2$.
The exact solutions are:

$$x = e^{1-t}$$
$$y = 2e^{1-t}. \quad (31)$$

Discussion of Result

Tables 1-16, we present the comparative results of the numerical problems solved. The Adams-Bashforth-Moulton method starting with the Runge-Kutta method converges to the exact solution faster than when starting with the Euler method, using step sizes of 0.1 and 0.05.
Tables 17-32, we compare the Adams-Bashforth-Moulton method of order 4 and order 5 (both starting with Runge-Kutta 4). It is noteworthy that the order 4 method yields more accurate results than the order 5 method in solving second-order differential equations. On the other hand, the Adams-Bashforth-Moulton method of order 5 yields more accurate results than the order 4 method in solving coupled differential equations.

Table 1: Numerical solution to problem 1 (y) and its error estimation when h=0.1

n	x	Y	ABM(E)	Error	ABM(RK)	Error
0	1	2				
1	1.1	2.27951				
2	1.2	2.76032				
3	1.3	3.51593				
4	1.4	4.63424	4.052998522	0.581241478	4.634197794	4.22057E-05
5	1.5	6.21875	5.484852621	0.733897379	6.21872293	2.70699E-05
6	1.6	8.38976	7.475295815	0.914464185	8.389745455	1.45453E-05
7	1.7	11.28557	10.16347679	1.122093213	11.28556542	4.57922E-06
8	1.8	15.06368	13.70137318	1.362306821	15.06368273	2.72682E-06
9	1.9	19.90199	18.26130319	1.640686813	19.90199716	7.15796E-06
10	2	26	24.12794949	1.872050506	26.00000855	8.55405E-06
11	2.1	33.58001	31.39776574	2.18224426	33.58001678	6.77607E-06
12	2.2	42.88832	40.36489426	2.523425738	42.88832166	1.65659E-06
13	2.3	54.19643	51.29930581	2.897124195	54.19642303	6.96712E-06
14	2.4	67.80224	64.49747865	3.304761352	67.80222075	1.925E-05
15	2.5	84.03125	80.28343139	3.747818612	84.03121465	3.53516E-05
16	2.6	103.23776	99.00997286	4.227787141	103.2377046	5.54324E-05
17	2.7	125.80607	121.0599347	4.746135265	125.8059903	7.96507E-05
18	2.8	152.15168	146.8473421	5.304337883	152.1515718	0.000108166
19	2.9	182.72249	176.818616	5.903873969	182.7223489	0.000141137
20	3	218	211.4537811	6.546218886	217.9998213	0.000178723

Table 2: Numerical solution to problem 1 (z) and its error estimation when h=0.1

N	x	Z	ABM(E)	Error	ABM(RK)	Error
0	1	2				
1	1.1	3.6905				
2	1.2	6.048				
3	1.3	9.2105				
4	1.4	13.328	11.87803072	1.449969276	13.32784377	0.000156226
5	1.5	18.5625	16.89076151	1.671738493	18.56232082	0.000179177
6	1.6	25.088	23.15803344	1.929966562	25.0877965	0.000203497
7	1.7	33.0905	30.85958171	2.230918295	33.09027006	0.000229943
8	1.8	42.768	40.18469386	2.583306136	42.76774213	0.000257868
9	1.9	54.3305	51.33485742	2.995642584	54.33021277	0.000287228
10	2	68	65.04784858	2.952151418	67.99968173	0.00031827
11	2.1	84.0105	80.75542169	3.255078313	84.01014908	0.000350917
12	2.2	102.608	99.03568944	3.572310558	102.6076149	0.00038512
13	2.3	124.0505	120.1460938	3.904406155	124.0500791	0.000420924
14	2.4	148.608	144.3566379	4.251362138	148.6075417	0.000458327
15	2.5	176.5625	171.9494891	4.613010889	176.5620027	0.000497315
16	2.6	208.208	203.2185801	4.989419875	208.2074621	0.000537895
17	2.7	243.8505	238.4698895	5.380610547	243.8499199	0.000580068
18	2.8	283.808	278.0214447	5.786555278	283.8073762	0.000623832
19	2.9	328.4105	322.2032408	6.20725918	328.4098308	0.000669187
20	3	378	371.3572723	6.642727679	377.9992839	0.000716134

Table 3: Numerical solution to problem 2 (Y) and its error estimation when h=0.1

n	x	y	ABM(E)	Error	ABM(RK)	Error
0	0.0	2.0				
1	0.1	1.222104137				
2	0.2	1.497737046				
3	0.3	1.843178201				
4	0.4	2.278311393	2.1344375	0.143873893	2.278291817	1.95753E-05
5	0.5	2.827422743	2.645702775	0.181719968	2.827394368	2.8375E-05
6	0.6	3.520175384	3.290192647	0.229982737	3.520131038	4.43465E-05
7	0.7	4.39279995	4.101924984	0.290874966	4.39272566	7.429E-05
8	0.8	5.489548637	5.120108081	0.369440555	5.489424283	0.000124353
9	0.9	6.864471197	6.392039797	0.472431399	6.86427166	0.000199537
10	1	8.583584148	8.037616616	0.545967533	8.583275241	0.000308908
11	1.1	10.72752025	10.053928	0.673592251	10.72705528	0.000464968
12	1.2	13.39476457	12.56529735	0.829467219	13.39408205	0.000682522
13	1.3	16.70560705	15.68570567	1.01990138	16.70462624	0.000980812
14	1.4	20.80697016	19.55442414	1.252546015	20.80558488	0.001385274
15	1.5	25.87830538	24.34157126	1.53673412	25.87637704	0.001928348
16	1.6	32.1387953	30.25489756	1.88389774	32.13614389	0.002651406
17	1.7	39.85615007	37.54815081	2.307999256	39.85254252	0.003607555
18	1.8	49.35735167	46.53127027	2.826081399	49.35248703	0.004864641
19	1.9	61.04177674	57.58280741	3.458969327	61.03526773	0.00650901
20	2	75.39722505	71.1651172	4.232107849	75.38857454	0.008650509

Table 4: Numerical solution to problem 2 (z) and its error estimation when h=0.1

n	x	z	ABM(E)	Error	ABM(RK)	Error
0	0.0	2.0				
1	0.1	2.464208274				
2	0.2	3.075474093				
3	0.3	3.866356401				
4	0.4	4.876622785	4.528902	0.347720785	4.876583635	3.91505E-05
5	0.5	6.154845485	5.7262484	0.428597085	6.154708875	0.00013661
6	0.6	7.760350768	7.22233808	0.538012688	7.760085116	0.000265653
7	0.7	9.765599901	9.077757696	0.687842205	9.765180994	0.000418907
8	0.8	12.25909727	11.36437324	0.894724038	12.25845279	0.000644479
9	0.9	15.34894239	14.16842388	1.180518511	15.34798515	0.000957239
10	1	19.1671683	18.01468102	1.152487274	19.16580228	0.001366021
11	1.1	23.8750405	22.46681839	1.408222106	23.87313212	0.001908376
12	1.2	29.66952914	27.94924043	1.720288707	29.66689997	0.002629171
13	1.3	36.79121411	34.68986814	2.101345969	36.78763994	0.00357416
14	1.4	45.53394031	42.96688138	2.567058934	45.52913366	0.00480665
15	1.5	56.25661077	53.12062402	3.135986753	56.25019976	0.006411005
16	1.6	69.39759059	65.56670529	3.830885306	69.38909987	0.008490725
17	1.7	85.49230014	80.81249829	4.679801856	85.48112386	0.01117628
18	1.8	105.1947033	99.47782183	5.716881505	105.1800686	0.014634777
19	1.9	129.3035535	122.3197993	6.983754217	129.2844762	0.019077243
20	2	158.7944501	150.2630934	8.531356656	158.7696808	0.024769347

Table 5: Numerical solution to problem 3 (x) and its error estimation when h=0.1

n	t	x	ABM(E)	Error	ABM(RK)	Error
0	0.0	2.000000				
1	0.1	2.845478				
2	0.2	3.805154				
3	0.3	4.917545				
4	0.4	6.227294	5.375105	0.852189	6.227262	0.000032
5	0.5	7.786966	6.771858	1.015108	7.786854	0.000112
6	0.6	9.659157	8.434833	1.224324	9.658923	0.000233
7	0.7	11.919003	10.433371	1.485632	11.918625	0.000378
8	0.8	14.657201	12.851039	1.806161	14.656596	0.000605
9	0.9	17.983644	15.788469	2.195174	17.982730	0.000913
10	1.0	22.031833	19.355185	2.676648	22.030505	0.001328
11	1.1	26.964237	23.701510	3.262727	26.962370	0.001867
12	1.2	32.978811	28.996255	3.982557	32.976214	0.002597
13	1.3	40.316941	35.458443	4.858497	40.313403	0.003537
14	1.4	49.273130	43.338732	5.934398	49.268349	0.004781
15	1.5	60.206824	52.963475	7.243349	60.200445	0.006378
16	1.6	73.556828	64.707977	8.848852	73.548357	0.008472
17	1.7	89.858927	79.055603	10.803324	89.847780	0.011147
18	1.8	109.767380	96.568335	13.199045	109.752758	0.014622
19	1.9	134.081183	117.964745	16.116438	134.062133	0.019050
20	2.0	163.776135	144.085621	19.690513	163.751373	0.024761

Table 6: Numerical solution to problem 3 (y) and its error estimation when h=0.1

n	t	y	ABM(E)	Error	ABM(RK)	Error
0	0.0	2.000000				
1	0.1	2.040134				
2	0.2	2.162145				
3	0.3	2.370930				
4	0.4	2.674870	2.452168	0.222702	2.674850	0.000020
5	0.5	3.086161	2.796026	0.290135	3.086091	0.000070
6	0.6	3.621311	3.253882	0.367429	3.621191	0.000121
7	0.7	4.301797	3.838961	0.462836	4.301616	0.000181
8	0.8	5.154929	4.580853	0.574076	5.154675	0.000254
9	0.9	6.214946	5.504534	0.710412	6.214584	0.000362
10	1.0	7.524391	6.651348	0.873044	7.523899	0.000492
11	1.1	9.135817	8.061926	1.073891	9.135140	0.000677
12	1.2	11.113894	9.799836	1.314059	11.112987	0.000908
13	1.3	13.538012	11.926977	1.611034	13.536784	0.001227
14	1.4	16.505457	14.537140	1.968317	16.503830	0.001626
15	1.5	20.135324	17.725640	2.409684	20.133156	0.002168
16	1.6	24.573292	21.631269	2.942024	24.570445	0.002847
17	1.7	29.997473	26.398401	3.599073	29.993721	0.003752
18	1.8	36.625558	32.232268	4.393290	36.620670	0.004888
19	1.9	44.723555	39.351294	5.372261	44.717173	0.006382
20	2.0	54.616466	48.058913	6.557552	54.608206	0.008260

Table 7: Numerical solution to problem 4 (x) and its error estimation when h=0.1

n	t	x	ABM(E)	Error	ABM(RK)	Error
0	1.0	1.000000000				
1	1.1	0.904837418				
2	1.2	0.818730753				
3	1.3	0.740818221				
4	1.4	0.670320046	0.659669531	0.010650515	0.670319147	8.98796E-07
5	1.5	0.606530660	0.596876250	0.009654410	0.606528663	1.99693E-06
6	1.6	0.548811636	0.540094533	0.008717103	0.548807882	3.75459E-06
7	1.7	0.496585304	0.488675174	0.007910130	0.496577858	7.44588E-06
8	1.8	0.449328964	0.442184173	0.007144792	0.449313931	1.50328E-05
9	1.9	0.406569660	0.400093162	0.006476498	0.406538392	3.12676E-05
10	2.0	0.367879441	0.362028416	0.005851025	0.367812943	6.6498E-05
11	2.1	0.332871084	0.327569266	0.005301817	0.332727426	0.000143658
12	2.2	0.301194212	0.296402599	0.004791613	0.300881692	0.000312520
13	2.3	0.272531793	0.268191194	0.004340599	0.271849590	0.000682203
14	2.4	0.246596964	0.242673059	0.003923905	0.245105253	0.001491711
15	2.5	0.223130160	0.219576444	0.003553716	0.219865500	0.003264660
16	2.6	0.201896518	0.198683290	0.003213228	0.194748925	0.007147593
17	2.7	0.182683524	0.179773972	0.002909552	0.167032007	0.015651517
18	2.8	0.165298888	0.162667675	0.002631213	0.131023142	0.034275746
19	2.9	0.149568619	0.147186422	0.002382197	0.074504305	0.075064315
20	3.0	0.135335283	0.133180700	0.002154583	0.029059216	0.164394500

Table 8: Numerical solution to problem 4 (y) and its error estimation when h=0.1

n	t	y	ABM(E)	Error	ABM(RK)	Error
0	1.0	2.000000000				
1	1.1	1.809674836				
2	1.2	1.637461506				
3	1.3	1.481636441				
4	1.4	1.340640092	1.319339063	0.02130103	1.340640374	2.82408E-07
5	1.5	1.213061319	1.193752500	0.01930882	1.213061256	6.3203E-08
6	1.6	1.097623272	1.080189066	0.01743421	1.097624026	7.53577E-07
7	1.7	0.993170608	0.977350348	0.01582026	0.993174128	3.52043E-06
8	1.8	0.898657928	0.884368345	0.01428958	0.898668492	1.05636E-05
9	1.9	0.813139319	0.800186324	0.01295300	0.813165575	2.6256E-05
10	2.0	0.735758882	0.724056832	0.01170205	0.735820075	6.11926E-05
11	2.1	0.665742167	0.655138532	0.01060364	0.665880240	0.000138072
12	2.2	0.602388424	0.592805199	0.00958323	0.602695249	0.000306826
13	2.3	0.545063586	0.536382389	0.00868120	0.545740002	0.000676416
14	2.4	0.493193928	0.485346118	0.00784781	0.494679873	0.001485945
15	2.5	0.446260320	0.439152888	0.00710743	0.449519247	0.003258927
16	2.6	0.403793036	0.397366580	0.00642646	0.410935004	0.007141969
17	2.7	0.365367048	0.359547944	0.00581910	0.381013057	0.015646009
18	2.8	0.330597776	0.325335351	0.00526243	0.364868178	0.034270401
19	2.9	0.299137238	0.294372844	0.00476440	0.374196376	0.075059138
20	3.0	0.270670566	0.266361400	0.00430917	0.435060085	0.164389519

Table 9: Numerical solution to problem 1 (y) and its error estimation when h=0.1

n	x	y	ABM4	Error	ABM5	Error
0	1.0	2.0				
1	1.1	2.27951				
2	1.2	2.76032				
3	1.3	3.51593				
4	1.4	4.63424	4.634197794	4.22057E-05		
5	1.5	6.21875	6.21872293	2.70699E-05	6.218639902	0.000110098
6	1.6	8.38976	8.389745455	1.45453E-05	8.389627064	0.000132936
7	1.7	11.28557	11.28556542	4.57922E-06	11.28541062	0.000159381
8	1.8	15.06368	15.06368273	2.72682E-06	15.063491	0.000189003
9	1.9	19.90199	19.90199716	7.15796E-06	19.90176799	0.000222007
10	2.0	26.00000000	26.00000855	8.55405E-06	25.99974124	0.000258761
11	2.1	33.58001000	33.58001678	6.77607E-06	33.57971062	0.000299384
12	2.2	42.88832000	42.88832166	1.65659E-06	42.88797599	0.000344013
13	2.3	54.19643000	54.19642303	6.96712E-06	54.19603708	0.000392924
14	2.4	67.80224000	67.80222075	1.925E-05	67.80179371	0.000446286
15	2.5	84.03125000	84.03121465	3.53516E-05	84.03074574	0.00050426
16	2.6	103.23776000	103.2377046	5.54324E-05	103.2371929	0.000567076
17	2.7	125.80607000	125.8059903	7.96507E-05	125.8054351	0.000634919
18	2.8	152.15168000	152.1515718	0.000108166	152.150972	0.000707969
19	2.9	182.72249000	182.7223489	0.000141137	182.7217036	0.000786432
20	3.0	218.00000000	217.9998213	0.000178723	217.9991295	0.000870498

Table 10: Numerical solution to problem 1 (z) and its error estimation when h=0.1

n	x	z	ABM4	Error	ABM5	Error
0	1.0	2.0				
1	1.1	3.6905				
2	1.2	6.048				
3	1.3	9.2105				
4	1.4	13.328	13.32784377	0.000156226		
5	1.5	18.5625	18.56232082	0.000179177	18.56228233	0.000217667
6	1.6	25.088	25.0877965	0.000203497	25.08775275	0.000247247
7	1.7	33.0905	33.09027006	0.000229943	33.09022119	0.000278812
8	1.8	42.768	42.76774213	0.000257868	42.76768686	0.000313145
9	1.9	54.3305	54.33021277	0.000287228	54.33015123	0.000348771
10	2.0	68.00000000	67.99968173	0.00031827	67.99961379	0.000386206
11	2.1	84.01050000	84.01014908	0.000350917	84.01007398	0.000426017
12	2.2	102.60800000	102.6076149	0.00038512	102.6075325	0.000467537
13	2.3	124.05050000	124.0500791	0.000420924	124.0499891	0.000510895
14	2.4	148.60800000	148.6075417	0.000458327	148.6074436	0.00055637
15	2.5	176.56250000	176.5620027	0.000497315	176.5618963	0.000603703
16	2.6	208.20800000	208.2074621	0.000537895	208.2073471	0.00065292
17	2.7	243.85050000	243.8499199	0.000580068	243.8497959	0.000704139
18	2.8	283.80800000	283.8073762	0.000623832	283.8072427	0.000757267
19	2.9	328.41050000	328.4098308	0.000669187	328.4096877	0.000812307
20	3.0	378.00000000	377.9992839	0.000716134	377.9991307	0.000869303

Table 11; Numerical solution to problem 2 (y) and its error estimation when h=0.1

n	x	y	ABM4	Error	ABM5	Error
0	0	2				
1	0.1	1.222104137				
2	0.2	1.497737046				
3	0.3	1.843178201				
4	0.4	2.278311393	2.278291817	1.95753E-05		
5	0.5	2.827422743	2.827394368	2.8375E-05	2.779084664	0.048338079
6	0.6	3.520175384	3.520131038	4.43465E-05	3.445682634	0.07449275
7	0.7	4.39279995	4.39272566	7.429E-05	4.29785045	0.0949495
8	0.8	5.489548637	5.489424283	0.000124353	5.352325247	0.137223389
9	0.9	6.864471197	6.86427166	0.000199537	6.69083722	0.173633977
10	1.0	8.583584148	8.583275241	0.000308908	8.590618055	0.007033906
11	1.1	10.72752025	10.72705528	0.000464968	10.73892619	0.011405942
12	1.2	13.39476457	13.39408205	0.000682522	13.41151495	0.016750376
13	1.3	16.70560705	16.70462624	0.000980812	16.72908170	0.023474646
14	1.4	20.80697016	20.80558488	0.001385274	20.83843552	0.031465361
15	1.5	25.87830538	25.87637704	0.001928348	25.91947615	0.041170763
16	1.6	32.13879530	32.13614389	0.002651406	32.19200892	0.053213629
17	1.7	39.85615007	39.85254252	0.003607555	39.92397673	0.067826657
18	1.8	49.35735167	49.35248703	0.004864641	49.44289323	0.085541567
19	1.9	61.04177674	61.03526773	0.006509010	61.14906888	0.107292136
20	2.0	75.39722505	75.38857454	0.008650509	75.53107166	0.133846611

Table 12: Numerical solution to problem 2 (z) and error estimation when h=0.1

n	x	z	ABM4	Error	ABM5	Error
0	0	2				
1	0.1	2.464208274				
2	0.2	3.075474093				
3	0.3	3.866356401				
4	0.4	4.876622785	4.876583635	3.91505 E-05		
5	0.5	6.154845485	6.154708875	0.00013661	6.154770941	7.45444E-05
6	0.6	7.760350768	7.760085116	0.000265653	7.76023703	0.000113738
7	0.7	9.765599901	9.765180994	0.000418907	9.765438215	0.000161685
8	0.8	12.25909727	12.25845279	0.000644479	12.28883882	0.029741543
9	0.9	15.34894239	15.34798515	0.000957239	15.38068905	0.031746661
10	1.0	19.1671683	19.16580228	0.001366021	19.20751864	0.040350343
11	1.1	23.8750405	23.87313212	0.001908376	23.92464323	0.049602735
12	1.2	29.66952914	29.66689997	0.002629171	29.72805564	0.058526502
13	1.3	36.79121411	36.78763994	0.003574160	36.86359668	0.072382574
14	1.4	45.53394031	45.52913366	0.004806650	45.62343382	0.089493509
15	1.5	56.25661077	56.25019976	0.006411005	56.36435936	0.107748586
16	1.6	69.39759059	69.38909987	0.008490725	69.52895547	0.131364880
17	1.7	85.49230014	85.48112386	0.011176280	85.65382953	0.161529389
18	1.8	105.1947033	105.1800686	0.014634777	105.3911480	0.196444718
19	1.9	129.3035535	129.2844762	0.019077243	129.5426175	0.239063996
20	2.0	158.7944501	158.7696808	0.024769347	159.0870573	0.292607179

Table 13: Numerical solution to problem 3 (x) and its error estimation when h=0.1

n	t	x	ABM(E)	Error	ABM(RK)	Error
0	0.0	2.000000				
1	0.1	2.845478				
2	0.2	3.805154				
3	0.3	4.917545				
4	0.4	6.227294	5.375105	0.852189	6.227262	0.000032
5	0.5	7.786966	6.771858	1.015108	7.786854	0.000112
6	0.6	9.659157	8.434833	1.224324	9.658923	0.000233
7	0.7	11.919003	10.433371	1.485632	11.918625	0.000378
8	0.8	14.657201	12.851039	1.806161	14.656596	0.000605
9	0.9	17.983644	15.788469	2.195174	17.982730	0.000913
10	1.0	22.031833	19.355185	2.676648	22.030505	0.001328
11	1.1	26.964237	23.701510	3.262727	26.962370	0.001867
12	1.2	32.978811	28.996255	3.982557	32.976214	0.002597
13	1.3	40.316941	35.458443	4.858497	40.313403	0.003537
14	1.4	49.273130	43.338732	5.934398	49.268349	0.004781
15	1.5	60.206824	52.963475	7.243349	60.200445	0.006378
16	1.6	73.556828	64.707977	8.848852	73.548357	0.008472
17	1.7	89.858927	79.055603	10.803324	89.847780	0.011147
18	1.8	109.767380	96.568335	13.199045	109.752758	0.014622
19	1.9	134.081183	117.964745	16.116438	134.062133	0.019050
20	2.0	163.776135	144.085621	19.690513	163.751373	0.024761

Table 14: Numerical solution to problem 3 (y) and its error estimation when h=0.1

n	t	y	ABM(E)	Error	ABM(RK)	Error
0	0.0	2.000000				
1	0.1	2.040134				
2	0.2	2.162145				
3	0.3	2.370930				
4	0.4	2.674870	2.452168	0.222702	2.674850	0.000020
5	0.5	3.086161	2.796026	0.290135	3.086091	0.000070
6	0.6	3.621311	3.253882	0.367429	3.621191	0.000121
7	0.7	4.301797	3.838961	0.462836	4.301616	0.000181
8	0.8	5.154929	4.580853	0.574076	5.154675	0.000254
9	0.9	6.214946	5.504534	0.710412	6.214584	0.000362
10	1.0	7.524391	6.651348	0.873044	7.523899	0.000492
11	1.1	9.135817	8.061926	1.073891	9.135140	0.000677
12	1.2	11.113894	9.799836	1.314059	11.112987	0.000908
13	1.3	13.538012	11.926977	1.611034	13.536784	0.001227
14	1.4	16.505457	14.537140	1.968317	16.503830	0.001626
15	1.5	20.135324	17.725640	2.409684	20.133156	0.002168
16	1.6	24.573292	21.631269	2.942024	24.570445	0.002847
17	1.7	29.997473	26.398401	3.599073	29.993721	0.003752
18	1.8	36.625558	32.232268	4.393290	36.620670	0.004888
19	1.9	44.723555	39.351294	5.372261	44.717173	0.006382
20	2.0	54.616466	48.058913	6.557552	54.608206	0.008260

Table 15: Numerical solution to problem 4 (x) and its error estimation when h=0.1

n	t	x	ABM(E)	Error	ABM(RK)	Error
0	1.0	1.000000000				
1	1.1	0.904837418				
2	1.2	0.818730753				
3	1.3	0.740818221				
4	1.4	0.670320046	0.659669531	0.010650515	0.670319147	8.98796E-07
5	1.5	0.606530660	0.596876250	0.009654410	0.606528663	1.99693E-06
6	1.6	0.548811636	0.540094533	0.008717103	0.548807882	3.75459E-06
7	1.7	0.496585304	0.488675174	0.007910130	0.496577858	7.44588E-06
8	1.8	0.449328964	0.442184173	0.007144792	0.449313931	1.50328E-05
9	1.9	0.406569660	0.400093162	0.006476498	0.406538392	3.12676E-05
10	2.0	0.367879441	0.362028416	0.005851025	0.367812943	6.6498E-05
11	2.1	0.332871084	0.327569266	0.005301817	0.332727426	0.000143658
12	2.2	0.301194212	0.296402599	0.004791613	0.300881692	0.000312520
13	2.3	0.272531793	0.268191194	0.004340599	0.271849590	0.000682203
14	2.4	0.246596964	0.242673059	0.003923905	0.245105253	0.001491711
15	2.5	0.223130160	0.219576444	0.003553716	0.219865500	0.003264660
16	2.6	0.201896518	0.198683290	0.003213228	0.194748925	0.007147593
17	2.7	0.182683524	0.179773972	0.002909552	0.167032007	0.015651517
18	2.8	0.165298888	0.162667675	0.002631213	0.131023142	0.034275746
19	2.9	0.149568619	0.147186422	0.002382197	0.074504305	0.075064315
20	3.0	0.135335283	0.133180700	0.002154583	0.029059216	0.164394500

Table 16: Numerical solution to problem 4 (y) and its error estimation when h=0.1

n	t	y	ABM(E)	Error	ABM(RK)	Error
0	1.0	2.000000000				
1	1.1	1.809674836				
2	1.2	1.637461506				
3	1.3	1.481636441				
4	1.4	1.340640092	1.319339063	0.02130103	1.340640374	2.82408E-07
5	1.5	1.213061319	1.193752500	0.01930882	1.213061256	6.3203E-08
6	1.6	1.097623272	1.080189066	0.01743421	1.097624026	7.53577E-07
7	1.7	0.993170608	0.977350348	0.01582026	0.993174128	3.52043E-06
8	1.8	0.898657928	0.884368345	0.01428958	0.898668492	1.05636E-05
9	1.9	0.813139319	0.800186324	0.01295300	0.813165575	2.6256E-05
10	2.0	0.735758882	0.724056832	0.01170205	0.735820075	6.11926E-05
11	2.1	0.665742167	0.655138532	0.01060364	0.665880240	0.000138072
12	2.2	0.602388424	0.592805199	0.00958323	0.602695249	0.000306826
13	2.3	0.545063586	0.536382389	0.00868120	0.545740002	0.000676416
14	2.4	0.493193928	0.485346118	0.00784781	0.494679873	0.001485945
15	2.5	0.446260320	0.439152888	0.00710743	0.449519247	0.003258927
16	2.6	0.403793036	0.397366580	0.00642646	0.410935004	0.007141969
17	2.7	0.365367048	0.359547944	0.00581910	0.381013057	0.015646009
18	2.8	0.330597776	0.325335351	0.00526243	0.364868178	0.034270401
19	2.9	0.299137238	0.294372844	0.00476440	0.374196376	0.075059138
20	3.0	0.270670566	0.266361400	0.00430917	0.435060085	0.164389519

Table 17: Numerical solution to problem 1 (y) and its error estimation when h=0.05

n	x	y	ABM(E)	Error	ABM(RK)	Error
0	1	2				
0.5	1.05	2.118656563				
1	1.1	2.27951				
1.5	1.15	2.490482188				
2	1.2	2.76032	2.667750778	0.092569222	2.760350287	3.02869E-05
2.5	1.25	3.098632813	2.999867656	0.098765157	3.098687963	5.51507E-05
3	1.3	3.51593	3.398176337	0.117753663	3.516039131	0.000109131
3.5	1.35	4.023658438	3.886434236	0.137224202	4.023810376	0.000151939
4	1.4	4.63424	4.47445936	0.15978064	4.634434267	0.000194267
4.5	1.45	5.361109063	5.175102427	0.186006635	5.361355007	0.000245945
5	1.5	6.21875	6.002228165	0.216521835	6.219050038	0.000300038
5.5	1.55	7.222734688	6.970752757	0.25198193	7.223091383	0.000356695
6	1.6	8.38976	8.096681301	0.293078699	8.390177868	0.000417868
6.5	1.65	9.737685313	9.397145261	0.340540052	9.738168202	0.000482889
7	1.7	11.28557	10.89043993	0.395130074	11.28612172	0.000551717
7.5	1.75	13.05371094	12.59606187	0.457649063	13.05433568	0.000624742
8	1.8	15.06368	14.53474643	0.528933567	15.06438202	0.000702016
8.5	1.85	17.33836156	16.72850514	0.609856419	17.33914518	0.000783621
9	1.9	19.90199	19.20066323	0.701326768	19.90285971	0.000869712
9.5	1.95	22.78018719	21.97589708	0.804290112	22.78114759	0.000960404
10	2	26	25.08027167	0.919728328	26.00105581	0.001055814
10.5	2.05	29.58993781	28.54127811	1.048659698	29.59109388	0.001156065
11	2.1	33.58001	32.38787106	1.192138938	33.58127128	0.001261279
11.5	2.15	38.00176344	36.65050622	1.351257222	38.00313501	0.001371576
12	2.2	42.88832	41.36117779	1.527142205	42.88980708	0.001487076
12.5	2.25	48.27441406	46.55345601	1.72095805	48.27602196	0.001607902
13	2.3	54.19643	52.26252456	1.933905445	54.19816417	0.001734174
13.5	2.35	60.69243969	58.52521806	2.167221629	60.6943057	0.001866014
14	2.4	67.80224	65.38005959	2.422180409	67.80424354	0.002003541
14.5	2.45	75.56739031	72.86729813	2.70009218	75.56953719	0.002146878
15	2.5	84.03125	81.02894605	3.002303947	84.03354615	0.002296145
15.5	2.55	93.23901594	89.9088166	3.330199336	93.2414674	0.002451464
16	2.6	103.23776	99.55256138	3.685198617	103.240373	0.002612955
16.5	2.65	114.0764666	110.0077078	4.068758719	114.0792473	0.002780739
17	2.7	125.80607	121.3236968	4.482373245	125.8090249	0.002954937
17.5	2.75	138.4794922	133.5519197	4.927572487	138.4826279	0.003135671
18	2.8	152.15168	146.7457566	5.405923443	152.1550031	0.003323062
18.5	2.85	166.8796428	160.960613	5.919029829	166.88316	0.00351723
19	2.9	182.72249	176.2539579	6.468532093	182.7262083	0.003718296
19.5	2.95	199.7414684	192.685361	7.056107429	199.7453948	0.003926382
20	3	218	210.3165302	7.68346979	218.0041416	0.004141609

Table 18: Numerical solution to problem 1 (z) and its error estimation when h=0.05

n	x	z	ABM(E)	Error	ABM(RK)	Error
0	1	2				
0.5	1.05	2.77003125				
1	1.1	3.6905				
1.5	1.15	4.77753125				
2	1.2	6.048	5.766890098	0.281109902	6.048837745	0.000837745
2.5	1.25	7.51953125	7.2027741	0.31675715	7.520254799	0.000723549
3	1.3	9.2105	8.848217168	0.362282832	9.211265859	0.000765859
3.5	1.35	11.14003125	10.72162733	0.418403925	11.14094426	0.000913012
4	1.4	13.328	12.84216139	0.485838607	13.32895385	0.000953847
4.5	1.45	15.79503125	15.22972508	0.565306174	15.79604046	0.001009213
5	1.5	18.5625	17.90497304	0.657526961	18.56359176	0.001091755
5.5	1.55	21.65253125	20.88930897	0.763222277	21.6536956	0.001164345
6	1.6	25.088	24.20488566	0.883114343	25.08923819	0.001238193
6.5	1.65	28.89253125	27.87460502	1.017926233	28.89384935	0.001318101
7	1.7	33.0905	31.92211817	1.168381825	33.09189928	0.001399282
7.5	1.75	37.70703125	36.37182549	1.335205757	37.70851371	0.001482459
8	1.8	42.768	41.24887662	1.519123377	42.7695685	0.001568497
8.5	1.85	48.30003125	46.57917054	1.720860712	48.30168813	0.00165688
9	1.9	54.3305	52.38935557	1.941144427	54.33224761	0.001747612
9.5	1.95	60.88753125	58.70682946	2.180701794	60.88937206	0.00184081
10	2	68	65.55973934	2.440260659	68.00193643	0.001936426
10.5	2.05	75.69753125	72.97698183	2.720549417	75.6995657	0.002034454
11	2.1	84.0105	80.98820302	3.022296982	84.01263491	0.002134906
11.5	2.15	92.97003125	89.62379848	3.346232765	92.97226903	0.002237779
12	2.2	102.608	98.91491335	3.69308665	102.6103431	0.002343072
12.5	2.25	112.9570313	108.8934423	4.063588974	112.959482	0.002450785
13	2.3	124.0505	119.5920295	4.458470504	124.0530609	0.002560919
13.5	2.35	135.9225313	131.0440688	4.878462424	135.9252047	0.002673474
14	2.4	148.608	143.2837037	5.32429631	148.6107884	0.002788449
14.5	2.45	162.1425313	156.3458271	5.796704122	162.1454371	0.002905845
15	2.5	176.5625	170.2660818	6.296418181	176.5655257	0.003025661
15.5	2.55	191.9050313	185.0808601	6.824171162	191.9081791	0.003147898
16	2.6	208.208	200.8273039	7.380696072	208.2112726	0.003272555
16.5	2.65	225.5100313	217.543305	7.966726244	225.5134309	0.003399633
17	2.7	243.8505	235.2675047	8.582995324	243.8540291	0.003529131
17.5	2.75	263.2695312	254.039294	9.230237255	263.2731923	0.00366105
18	2.8	283.808	273.8988137	9.909186271	283.8117954	0.003795389
18.5	2.85	305.5075312	294.8869544	10.62057689	305.5114634	0.003932149
19	2.9	328.4105	317.0453561	11.36514388	328.4145713	0.004071329
19.5	2.95	352.5600312	340.416409	12.1436223	352.5642442	0.00421293
20	3	378	365.0432526	12.95674743	378.004357	0.004356952

Table 19: Numerical solution to problem 2 (y) and its error estimation when h=0.05

n	x	y	ABM(E)	Error	ABM(RK)	Error
0	0	2				
0.5	0.05	1.105256377				
1	0.1	1.222104137				
1.5	0.15	1.352288211				
2	0.2	1.497737046	1.473842578	0.023894468	1.497736811	2.35929E-07
2.5	0.25	1.660581906	1.635576008	0.025005898	1.660581905	6.85074E-10
3	0.3	1.843178201	1.813639908	0.029538293	1.84317847	2.69831E-07
3.5	0.35	2.048129061	2.013845344	0.034283717	2.048129598	5.36377E-07
4	0.4	2.278311393	2.238303761	0.040007631	2.278312198	8.05436E-07
4.5	0.45	2.536904667	2.489940458	0.046964209	2.536905754	1.08748E-06
5	0.5	2.827422743	2.771973261	0.055449481	2.827424117	1.37384E-06
5.5	0.55	3.153749036	3.087941783	0.065807253	3.153750697	1.66117E-06
6	0.6	3.520175384	3.441739594	0.07843579	3.520177334	1.94961E-06
6.5	0.65	3.931445001	3.837649624	0.093795378	3.931447237	2.23535E-06
7	0.7	4.39279995	4.280383094	0.112416856	4.392802464	2.51393E-06
7.5	0.75	4.910033606	4.775122349	0.134911257	4.910036387	2.78126E-06
8	0.8	5.489548637	5.327567966	0.16198067	5.489551668	3.03188E-06
8.5	0.85	6.138421088	5.943990583	0.194430504	6.138424347	3.25914E-06
9	0.9	6.864471197	6.631287899	0.233183297	6.864474652	3.45543E-06
9.5	0.95	7.676341663	7.397047385	0.279294279	7.676345275	3.61174E-06
10	1	8.583584148	8.249615256	0.333968892	8.583587866	3.71747E-06
10.5	1.05	9.596754869	9.198172352	0.398582517	9.596758629	3.76026E-06
11	1.1	10.72752025	10.25281759	0.474702654	10.72752397	3.72569E-06
11.5	1.15	11.98877368	11.4246598	0.564113883	11.98877728	3.59702E-06
12	1.2	13.39476457	12.72591866	0.668845909	13.39476793	3.35484E-06
12.5	1.25	14.96124094	14.17003585	0.791205092	14.96124392	2.97671E-06
13	1.3	16.70560705	15.77179719	0.933809861	16.70560949	2.43669E-06
13.5	1.35	18.64709759	17.54746711	1.099630481	18.64709929	1.70492E-06
14	1.4	20.80697016	19.51493645	1.292033707	20.8069709	7.47037E-07
14.5	1.45	23.20871805	21.69388516	1.514832889	23.20871758	4.76417E-07
15	1.5	25.87830538	24.10596119	1.772344196	25.87830337	2.01071E-06
15.5	1.55	28.84442692	26.77497725	2.069449669	28.84442301	3.9077E-06
16	1.6	32.1387953	29.72712736	2.411667934	32.13878907	6.2267E-06
16.5	1.65	35.79645838	32.99122492	2.805233463	35.79644935	9.0355E-06
17	1.7	39.85615007	36.59896467	3.257185403	39.85613766	1.24115E-05
17.5	1.75	44.36067794	40.58521083	3.775467108	44.3606615	1.64428E-05
18	1.8	49.35735167	44.98831405	4.36903762	49.35733044	2.12301E-05
18.5	1.85	54.89845654	49.85046002	5.04799652	54.89842965	2.68877E-05
19	1.9	61.04177674	55.21805303	5.82372371	61.04174319	3.35458E-05
19.5	1.95	67.85117366	61.14213778	6.70903588	67.85113231	4.13521E-05
20	2	75.39722505	67.67886344	7.71836161	75.39717458	5.04747E-05

Table 20: Numerical solution to problem 2 (z) and its error estimation when h=0.05

n	x	z	ABM(E)	Error	ABM(RK)	Error
0	0	2				
0.5	0.05	2.215512754				
1	0.1	2.464208274				
1.5	0.15	2.749576423				
2	0.2	3.075474093	3.01194875	0.063525343	3.075473621	4.71858E-07
2.5	0.25	3.446163812	3.372792375	0.073371437	3.446162758	1.05419E-06
3	0.3	3.866356401	3.779720363	0.086636039	3.86635486	1.5411E-06
3.5	0.35	4.341258122	4.237341149	0.103916974	4.341256151	1.97173E-06
4	0.4	4.876622785	4.750724014	0.125898772	4.876620148	2.63738E-06
4.5	0.45	5.478809333	5.325445165	0.153364168	5.478805923	3.41074E-06
5	0.5	6.154845485	5.967638431	0.187207054	6.154841218	4.26758E-06
5.5	0.55	6.912498072	6.684051025	0.228447047	6.912492783	5.28855E-06
6	0.6	7.760350768	7.482104877	0.278245891	7.760344284	6.48451E-06
6.5	0.65	8.707890003	8.369964115	0.337925888	8.707882139	7.86343E-06
7	0.7	9.765599901	9.356609276	0.408990624	9.765590438	9.46213E-06
7.5	0.75	10.94506721	10.45191895	0.493148257	10.9450559	1.13129E-05
8	0.8	12.25909727	11.6667596	0.592337674	12.25908383	1.34467E-05
8.5	0.85	13.72184218	13.01308431	0.708757866	13.72182627	1.59039E-05
9	0.9	15.34894239	14.50404149	0.844900903	15.34892366	1.87299E-05
9.5	0.95	17.15768333	16.15409439	1.003588938	17.15766135	2.19742E-05
10	1	19.1671683	17.97915258	1.188015719	19.1671426	2.56936E-05
10.5	1.05	21.39850974	19.99671659	1.401793152	21.39847979	2.99524E-05
11	1.1	23.8750405	22.22603699	1.649003504	23.87500568	3.48229E-05
11.5	1.15	26.62254736	24.68828944	1.93425792	26.62250698	4.03865E-05
12	1.2	29.66952914	27.40676714	2.262762004	29.66948241	4.67351E-05
12.5	1.25	33.04748188	30.4070926	2.64038928	33.04742791	5.39725E-05
13	1.3	36.79121411	33.71745061	3.073763493	36.79115189	6.22152E-05
13.5	1.35	40.93919517	37.36884442	3.570350751	40.93912358	7.15947E-05
14	1.4	45.53394031	41.39537762	4.138562697	45.53385805	8.2259E-05
14.5	1.45	50.62243611	45.83456413	4.787871981	50.62234173	9.43745E-05
15	1.5	56.25661077	50.72766929	5.528941479	56.25650264	0.000108129
15.5	1.55	62.49385384	56.12008497	6.373768875	62.49373011	0.000123732
16	1.6	69.39759059	62.06174222	7.335848375	69.39744917	0.000141421
16.5	1.65	77.03791676	68.60756519	8.430351574	77.0377553	0.000161463
17	1.7	85.49230014	75.81797046	9.674329685	85.49211599	0.000184155
17.5	1.75	94.84635588	83.75941625	11.08693962	94.84614604	0.000209835
18	1.8	105.1947033	92.50500663	12.6896967	105.1944645	0.00023888
18.5	1.85	116.6419131	102.135156	14.50675704	116.6416414	0.000271713
19	1.9	129.3035535	112.7383204	16.56523309	129.3032447	0.00030881
19.5	1.95	143.3073473	124.4118012	18.89554613	143.3069966	0.000350705
20	2	158.7944501	137.2626301	21.53182005	158.7940521	0.000397996

Table 21: Numerical solution to problem 3 (x) and its error estimation when h=0.05

n	t	x	ABM(E)	Error	ABM(RK)	Error
0	0	2				
0.5	0.05	2.410675336				
1	0.1	2.845477521				
1.5	0.15	3.308758202				
2	0.2	3.805154047	3.753015625	0.052138422	3.805153652	3.94753E-07
2.5	0.25	4.339633152	4.279950417	0.059682736	4.339632666	4.86645E-07
3	0.3	4.917544765	4.849739339	0.067805426	4.917544124	6.40922E-07
3.5	0.35	5.544672819	5.467947953	0.076724865	5.544672063	7.55462E-07
4	0.4	6.227293821	6.140885025	0.086408796	6.227292598	1.22382E-06
4.5	0.45	6.972239674	6.875347204	0.09689247	6.972237865	1.80898E-06
5	0.5	7.786966044	7.678571241	0.108394803	7.78696349	2.55468E-06
5.5	0.55	8.679626988	8.558660826	0.120966162	8.679623514	3.47455E-06
6	0.6	9.659156556	9.524403851	0.134752705	9.659151933	4.62319E-06
6.5	0.65	10.73535821	10.58547829	0.149879915	10.73535225	5.96346E-06
7	0.7	11.91900294	11.75248456	0.166518375	11.91899538	7.55928E-06
7.5	0.75	13.22193705	13.03712249	0.184814557	13.22192764	9.41351E-06
8	0.8	14.65720076	14.45223424	0.204966516	14.65718918	1.15742E-05
8.5	0.85	16.23915865	16.01199271	0.227165938	16.23914459	1.40624E-05
9	0.9	17.98364351	17.73200084	0.251642669	17.98362657	1.6934E-05
9.5	0.95	19.90811471	19.62947964	0.278635071	19.90809448	2.02259E-05
10	1	22.03183301	21.72341406	0.308418951	22.03180901	2.40015E-05
10.5	1.05	24.37605331	24.03476537	0.341287935	24.37602499	2.83168E-05
11	1.1	26.96423734	26.58666285	0.377574494	26.96420409	3.3248E-05
11.5	1.15	29.82228852	29.40464945	0.41763907	29.82224965	3.88711E-05
12	1.2	32.97881119	32.51692627	0.461884914	32.97876591	4.52807E-05
12.5	1.25	36.46539688	35.95464381	0.510753075	36.46534431	5.25767E-05
13	1.3	40.31694053	39.75220639	0.564734141	40.31687965	6.08773E-05
13.5	1.35	44.57198966	43.9476224	0.624367259	44.57191935	7.03116E-05
14	1.4	49.27313025	48.58288001	0.690250238	49.27304922	8.10285E-05
14.5	1.45	54.46741289	53.70437112	0.763041768	54.4673197	9.31929E-05
15	1.5	60.2068237	59.36335266	0.843471037	60.20671671	0.000106993
15.5	1.55	66.54880464	65.61646204	0.932342597	66.548682	0.000122637
16	1.6	73.55682839	72.52628202	1.030546369	73.55668802	0.000140364
16.5	1.65	81.30103359	80.16196861	1.139064987	81.30087316	0.000160438
17	1.7	89.85892687	88.59994198	1.258984888	89.85874371	0.000183159
17.5	1.75	99.31615849	97.92465232	1.391506176	99.31594963	0.000208862
18	1.8	109.7673796	108.2294242	1.537955449	109.7671417	0.000237925
18.5	1.85	121.3171896	119.6173911	1.699798422	121.3169188	0.00027077
19	1.9	134.0811827	132.2025276	1.87865512	134.0808748	0.000307874
19.5	1.95	148.1871054	146.1107897	2.076315681	148.1867556	0.000349769
20	2	163.7761345	161.4813759	2.294758599	163.7757374	0.000397054

Table 22: Numerical solution to problem 3 (y) and its error estimation when h=0.05

n	t	y	ABM(E)	Error	ABM(RK)	Error
0	0	2				
0.5	0.05	2.010008336				
1	0.1	2.040133511				
1.5	0.15	2.090677028				
2	0.2	2.162144744	2.130564583	0.03158016	2.162144528	2.15969E-07
2.5	0.25	2.25525193	2.222485139	0.032766792	2.255251031	8.99656E-07
3	0.3	2.370930436	2.336705824	0.034224613	2.370929041	1.39506E-06
3.5	0.35	2.510338011	2.474216216	0.036121795	2.51033616	1.85152E-06
4	0.4	2.674869893	2.636540572	0.038329321	2.674867625	2.26799E-06
4.5	0.45	2.866172771	2.82523995	0.04093282	2.866170059	2.71186E-06
5	0.5	3.08616127	3.042228302	0.043932968	3.086158164	3.10572E-06
5.5	0.55	3.337037108	3.289645964	0.047391144	3.337033564	3.54389E-06
6	0.6	3.621311135	3.57000475	0.051306385	3.621307148	3.98647E-06
6.5	0.65	3.941828461	3.886081024	0.055747437	3.94182398	4.48095E-06
7	0.7	4.301796931	4.241058933	0.060737998	4.301791918	5.01248E-06
7.5	0.75	4.70481923	4.63847609	0.066343141	4.704813609	5.62122E-06
8	0.8	5.154928942	5.082322467	0.072606476	5.154922642	6.30044E-06
8.5	0.85	5.656630916	5.577029534	0.079601382	5.656623833	7.08287E-06
9	0.9	6.214946353	6.127557179	0.087389173	6.21493838	7.97292E-06
9.5	0.95	6.835463062	6.739408441	0.09605462	6.835454062	8.99979E-06
10	1	7.524391382	7.418712288	0.105679094	7.524381206	1.01757E-05
10.5	1.05	8.288626341	8.172263091	0.116363249	8.28861481	1.15304E-05
11	1.1	9.135816658	9.007606134	0.128210524	9.135803574	1.30837E-05
11.5	1.15	10.0744413	9.933098974	0.141342325	10.07442643	1.48683E-05
12	1.2	11.11389433	10.9580065	0.155887836	11.11387742	1.69121E-05
12.5	1.25	12.26457896	12.09258451	0.171994452	12.26455971	1.92532E-05
13	1.3	13.53801161	13.34818967	0.189821943	13.53798968	2.19289E-05
13.5	1.35	14.94693724	14.73738733	0.209549911	14.94691225	2.49851E-05
14	1.4	16.50545683	16.27408195	0.231374887	16.50542836	2.84701E-05
14.5	1.45	18.22916859	17.97365252	0.255516071	18.22913615	3.24408E-05
15	1.5	20.13532399	19.85310949	0.282214499	20.13528703	3.69589E-05
15.5	1.55	22.24300048	21.9312626	0.311737883	22.24295839	4.20955E-05
16	1.6	24.5732924	24.22891106	0.344381345	24.57324447	4.79289E-05
16.5	1.65	27.14952209	26.76905016	0.380471931	27.14946754	5.45485E-05
17	1.7	29.99747332	29.57710269	0.420370632	29.99741126	6.20534E-05
17.5	1.75	33.14564934	32.68117234	0.464477004	33.14557879	7.05559E-05
18	1.8	36.62555817	36.11232581	0.513232357	36.62547798	8.01815E-05
18.5	1.85	40.47202789	39.90490306	0.567124825	40.47193682	9.10715E-05
19	1.9	44.72355527	44.09686154	0.626693727	44.72345188	0.000103384
19.5	1.95	49.42269102	48.73015563	0.692535385	49.42257372	0.000117297
20	2	54.61646567	53.85115691	0.765308757	54.61633266	0.00013301

Table 23: Numerical solution to problem 4 (x) and its error estimation when h=0.05

n	t	x	ABM(E)	Error	ABM(RK)	Error
0	1	1				
0.5	1.05	0.951229425				
1	1.1	0.904837418				
1.5	1.15	0.860707976				
2	1.2	0.818730753	0.693686133	0.12504462	0.818730806	5.31199E-08
2.5	1.25	0.778800783	0.659764922	0.119035861	0.778800856	7.28939E-08
3	1.3	0.740818221	0.627639211	0.11317901	0.740818271	5.08135E-08
3.5	1.35	0.70468809	0.596975386	0.107712704	0.704688154	6.41754E-08
4	1.4	0.670320046	0.56788282	0.102437226	0.670320175	1.28897E-07
4.5	1.45	0.637628152	0.540172902	0.09745525	0.637628352	2.00473E-07
5	1.5	0.60653066	0.513837323	0.092693337	0.60653095	2.90304E-07
5.5	1.55	0.57694981	0.488771989	0.088177821	0.576950261	4.50563E-07
6	1.6	0.548811636	0.464937308	0.083874328	0.548812341	7.0499E-07
6.5	1.65	0.522045777	0.442260204	0.079785573	0.522046849	1.07201E-06
7	1.7	0.496585304	0.420692003	0.075893301	0.49658692	1.6167E-06
7.5	1.75	0.472366553	0.400173961	0.072192592	0.472368995	2.44261E-06
8	1.8	0.449328964	0.380657626	0.068671338	0.449332643	3.67891E-06
8.5	1.85	0.427414932	0.362092499	0.065322433	0.427420449	5.51696E-06
9	1.9	0.40656966	0.344433171	0.062136489	0.406577918	8.25875E-06
9.5	1.95	0.386741023	0.327634881	0.059106143	0.386753377	1.23535E-05
10	2	0.367879441	0.311655983	0.056223458	0.367897901	1.84602E-05
10.5	2.05	0.349937749	0.296456308	0.053481441	0.349965313	2.75636E-05
11	2.1	0.332871084	0.281997976	0.050873108	0.332912222	4.11388E-05
11.5	2.15	0.316636769	0.268244758	0.048392011	0.316698152	6.13831E-05
12	2.2	0.301194212	0.255162309	0.046031903	0.301285781	9.15687E-05
12.5	2.25	0.286504797	0.242717889	0.043786908	0.286641374	0.000136577
13	2.3	0.272531793	0.230880397	0.041651396	0.272735481	0.000203688
13.5	2.35	0.259240261	0.219620222	0.039620038	0.259544016	0.000303755
14	2.4	0.246596964	0.208909215	0.037687749	0.247049927	0.000452963
14.5	2.45	0.234570288	0.198720589	0.035849699	0.235245729	0.000675441
15	2.5	0.22313016	0.189028869	0.034101291	0.224137331	0.001007171
15.5	2.55	0.212247974	0.17980982	0.032438154	0.213749777	0.001501803
16	2.6	0.201896518	0.171040389	0.030856129	0.20413585	0.002239332
16.5	2.65	0.192049909	0.162698648	0.029351261	0.195388948	0.003339039
17	2.7	0.182683524	0.154763739	0.027919785	0.187662301	0.004978777
17.5	2.75	0.173773943	0.14721582	0.026558123	0.181197681	0.007423738
18	2.8	0.165298888	0.140036018	0.02526287	0.176368229	0.011069341
18.5	2.85	0.157237166	0.133206378	0.024030788	0.17374235	0.016505183
19	2.9	0.149568619	0.126709825	0.022858794	0.174179016	0.024610396
19.5	2.95	0.142274072	0.120530112	0.02174396	0.178969895	0.036695823
20	3	0.135335283	0.114651787	0.020683496	0.190051304	0.05471602

Table 24: Numerical solution to problem 4 (y) and its error estimation when h=0.05

n	t	y	ABM(E)	Error	ABM(RK)	Error
0	1	2				
0.5	1.05	1.902458849				
1	1.1	1.809674836				
1.5	1.15	1.721415953				
2	1.2	1.637461506	1.387372266	0.250089241	1.637461491	1.54268E-08
2.5	1.25	1.557601566	1.319529844	0.238071722	1.557601492	7.399E-08
3	1.3	1.481636441	1.255278422	0.226358019	1.481636356	8.49416E-08
3.5	1.35	1.409376179	1.193950771	0.215425408	1.409376051	1.2877E-07
4	1.4	1.340640092	1.13576564	0.204874452	1.340639872	2.20269E-07
4.5	1.45	1.275256303	1.080345803	0.1949105	1.275255987	3.16323E-07
5	1.5	1.213061319	1.027674646	0.185386673	1.213060892	4.2782E-07
5.5	1.55	1.153899621	0.977543979	0.176355642	1.153899013	6.07491E-07
6	1.6	1.097623272	0.929874615	0.167748657	1.097622393	8.79026E-07
6.5	1.65	1.044091554	0.884520407	0.159571146	1.044090292	1.26116E-06
7	1.7	0.993170608	0.841384006	0.151786602	0.993168789	1.81905E-06
7.5	1.75	0.944733105	0.800347921	0.144385184	0.944730449	2.65643E-06
8	1.8	0.898657928	0.761315251	0.137342677	0.898654026	3.9026E-06
8.5	1.85	0.854829864	0.724184999	0.130644865	0.854824115	5.74905E-06
9	1.9	0.813139319	0.688866342	0.124272978	0.813130822	8.49788E-06
9.5	1.95	0.773482047	0.655269762	0.118212285	0.773469448	1.25984E-05
10	2	0.735758882	0.623311966	0.112446917	0.735740173	1.87098E-05
10.5	2.05	0.699875498	0.592912616	0.106962882	0.699847681	2.78168E-05
11	2.1	0.665742167	0.563995952	0.101746215	0.665700773	4.13947E-05
11.5	2.15	0.633273539	0.536489516	0.096784022	0.633211898	6.16408E-05
12	2.2	0.602388424	0.510324618	0.092063805	0.602296596	9.18275E-05
12.5	2.25	0.573009594	0.485435779	0.087573815	0.572872758	0.000136836
13	2.3	0.545063586	0.461760794	0.083302792	0.54485964	0.000203946
13.5	2.35	0.518480521	0.439240445	0.079240076	0.518176508	0.000304013
14	2.4	0.493193928	0.417818431	0.075375497	0.492740709	0.000453219
14.5	2.45	0.469140576	0.397441179	0.071699398	0.468464881	0.000675695
15	2.5	0.44626032	0.378057738	0.068202582	0.445252898	0.001007423
15.5	2.55	0.424495948	0.359619639	0.064876308	0.422993895	0.001502052
16	2.6	0.403793036	0.342080777	0.061712259	0.401553457	0.002239579
16.5	2.65	0.384099817	0.325397296	0.058702521	0.380760535	0.003339282
17	2.7	0.365367048	0.309527478	0.05583957	0.360388032	0.004979016
17.5	2.75	0.347547887	0.29443164	0.053116246	0.340123914	0.007423973
18	2.8	0.330597776	0.280072036	0.050525741	0.319528204	0.011069572
18.5	2.85	0.314474333	0.266412757	0.048061575	0.297968922	0.016505411
19	2.9	0.299137238	0.25341965	0.045717588	0.274526619	0.02461062
19.5	2.95	0.284548143	0.241060224	0.043487919	0.247852101	0.036696042
20	3	0.270670566	0.229303575	0.041366991	0.215954332	0.054716235

Table 25: Numerical solution to problem 1 (y) and its error estimation

n	x	y	ABM4	Error	ABM5	Error
0	1	2				
0.5	1.05	2.118657				
1	1.1	2.27951				
1.5	1.15	2.490482				
2	1.2	2.76032	2.76035	3.02869E-05		
2.5	1.25	3.098633	3.098688	5.51507E-05	3.098649	1.59467E-05
3	1.3	3.51593	3.516039	0.000109131	3.515966	3.56336E-05
3.5	1.35	4.023658	4.02381	0.000151939	4.023757	9.81536E-05
4	1.4	4.63424	4.634434	0.000194267	4.634367	0.000126698
4.5	1.45	5.361109	5.361355	0.000245945	5.361269	0.000159631
5	1.5	6.21875	6.21905	0.000300038	6.218966	0.000216342
5.5	1.55	7.222735	7.223091	0.000356695	7.223	0.000261911
6	1.6	8.38976	8.390178	0.000417868	8.39007	0.000310295
6.5	1.65	9.737685	9.738168	0.000482889	9.738054	0.000368978
7	1.7	11.28557	11.286122	0.000551717	11.286	0.000426455
7.5	1.75	13.053711	13.054336	0.000624742	13.054198	0.00048739
8	1.8	15.06368	15.064382	0.000702016	15.064234	0.000554145
8.5	1.85	17.338362	17.339145	0.000783621	17.338985	0.000623214
9	1.9	19.90199	19.90286	0.000869712	19.902686	0.000696147
9.5	1.95	22.780187	22.781148	0.000960404	22.780961	0.000773654
10	2	26	26.001056	0.001055814	26.000855	0.000854802
10.5	2.05	29.589938	29.591094	0.001156065	29.590878	0.000940137
11	2.1	33.58	33.581271	0.001261279	33.58104	0.001029902
11.5	2.15	38.001763	38.003135	0.001371576	38.002887	0.001123924
12	2.2	42.88832	42.889807	0.001487076	42.889542	0.001222437
12.5	2.25	48.274414	48.276022	0.001607902	48.27574	0.00132557
13	2.3	54.19643	54.198164	0.001734174	54.197863	0.001433359
13.5	2.35	60.69244	60.694306	0.001866014	60.693986	0.001545943
14	2.4	67.80224	67.804244	0.002003541	67.803903	0.00166343
14.5	2.45	75.56739	75.569537	0.002146878	75.569176	0.001785907
15	2.5	84.03125	84.033546	0.002296145	84.033163	0.001913489
15.5	2.55	93.239016	93.241467	0.002451464	93.241062	0.00204628
16	2.6	103.23776	103.240373	0.002612955	103.239944	0.002184379
16.5	2.65	114.076467	114.079247	0.002780739	114.078794	0.002327895
17	2.7	125.80607	125.809025	0.002954937	125.808547	0.002476929
17.5	2.75	138.479492	138.482628	0.003135671	138.482124	0.002631587
18	2.8	152.15168	152.155003	0.003323062	152.154472	0.002791972
18.5	2.85	166.879643	166.88316	0.00351723	166.882601	0.002958188
19	2.9	182.72249	182.726208	0.003718296	182.72562	0.00313034
19.5	2.95	199.741468	199.745395	0.003926382	199.744777	0.003308533
20	3	218	218.004142	0.004141609	218.003493	0.003492869

Table 26: Numerical solution to problem 1 (z) and its error estimation

n	x	z	ABM4	Error	ABM5	Error
0	1	2				
0.5	1.05	2.77003125				
1	1.1	3.6905				
1.5	1.15	4.77753125				
2	1.2	6.048	6.048837745	0.000837745		
2.5	1.25	7.51953125	7.520254799	0.000723549	7.520403581	0.000872331
3	1.3	9.2105	9.211265859	0.000765859	9.211084531	0.000584531
3.5	1.35	11.14003125	11.14094426	0.000913012	11.14074678	0.000715528
4	1.4	13.328	13.32895385	0.000953847	13.3289247	0.0009247
4.5	1.45	15.79503125	15.79604046	0.001009213	15.79586425	0.000833
5	1.5	18.5625	18.56359176	0.001091755	18.56341438	0.000914377
5.5	1.55	21.65253125	21.6536956	0.001164345	21.65356527	0.001034019
6	1.6	25.088	25.08923819	0.001238193	25.08905384	0.001053837
6.5	1.65	28.89253125	28.89384935	0.001318101	28.89365843	0.001127182
7	1.7	33.0905	33.09189928	0.001399282	33.09171385	0.001213851
7.5	1.75	37.70703125	37.70851371	0.001482459	37.70830326	0.001272013
8	1.8	42.768	42.7695685	0.001568497	42.76934769	0.001347692
8.5	1.85	48.30003125	48.30168813	0.00165688	48.30145965	0.001428395
9	1.9	54.3305	54.33224761	0.001747612	54.33200266	0.001502659
9.5	1.95	60.88753125	60.88937206	0.00184081	60.88911468	0.001583432
10	2	68	68.00193643	0.001936426	68.00166688	0.001666884
10.5	2.05	75.69753125	75.6995657	0.002034454	75.69928146	0.001750214
11	2.1	84.0105	84.01263491	0.002134906	84.01233684	0.001836843
11.5	2.15	92.97003125	92.97226903	0.002237779	92.97195688	0.001925631
12	2.2	102.608	102.6103431	0.002343072	102.610016	0.002015972
12.5	2.25	112.9570313	112.959482	0.002450785	112.95914	0.002108714
13	2.3	124.0505	124.0530609	0.002560919	124.0527035	0.002203534
13.5	2.35	135.9225313	135.9252047	0.002673474	135.9248316	0.002300319
14	2.4	148.608	148.6107884	0.002788449	148.6103993	0.002399265
14.5	2.45	162.1425313	162.1454371	0.002905845	162.1450315	0.002500286
15	2.5	176.5625	176.5655257	0.003025661	176.5651034	0.002603366
15.5	2.55	191.9050313	191.9081791	0.003147898	191.9077398	0.002708547
16	2.6	208.208	208.2112726	0.003272555	208.2108158	0.002815808
16.5	2.65	225.5100313	225.5134309	0.003399633	225.5129564	0.002925147
17	2.7	243.8505	243.8540291	0.003529131	243.8535366	0.003036572
17.5	2.75	263.2695312	263.2731923	0.00366105	263.2726813	0.003150079
18	2.8	283.808	283.8117954	0.003795389	283.8112657	0.003265669
18.5	2.85	305.5075312	305.5114634	0.003932149	305.5109146	0.003383341
19	2.9	328.4105	328.4145713	0.004071329	328.4140031	0.003503096
19.5	2.95	352.5600312	352.5642442	0.00421293	352.5636562	0.003624934
20	3	378	378.004357	0.004356952	378.0037489	0.003748855

Table 27: Numerical solution to problem 2 (y) and its error estimation

n	x	y	ABM4	Error	ABM5	Error
0	0	2				
0.5	0.05	1.105256377				
1	0.1	1.222104137				
1.5	0.15	1.352288211				
2	0.2	1.497737046	1.497736811	2.35929E-07		
2.5	0.25	1.660581906	1.660581905	6.85074E-10	1.660581162	7.44477E-07
3	0.3	1.843178201	1.84317847	2.69831E-07	1.843177393	8.07469E-07
3.5	0.35	2.048129061	2.048129598	5.36377E-07	2.048128182	8.7906E-07
4	0.4	2.278311393	2.278312198	8.05436E-07	2.278310429	9.639E-07
4.5	0.45	2.536904667	2.536905754	1.08748E-06	2.536903605	1.06165E-06
5	0.5	2.827422743	2.827424117	1.37384E-06	2.827421571	1.17174E-06
5.5	0.55	3.153749036	3.153750697	1.66117E-06	3.153747737	1.29911E-06
6	0.6	3.520175384	3.520177334	1.94961E-06	3.520173939	1.44536E-06
6.5	0.65	3.931445001	3.931447237	2.23535E-06	3.93144339	1.61159E-06
7	0.7	4.39279995	4.392802464	2.51393E-06	4.392798148	1.80186E-06
7.5	0.75	4.910033606	4.910036387	2.78126E-06	4.910031586	2.01922E-06
8	0.8	5.489548637	5.489551668	3.03188E-06	5.48954637	2.26653E-06
8.5	0.85	6.138421088	6.138424347	3.25914E-06	6.138418539	2.54826E-06
9	0.9	6.864471197	6.864474652	3.45543E-06	6.864468328	2.86898E-06
9.5	0.95	7.676341663	7.676345275	3.61174E-06	7.67633843	3.23338E-06
10	1	8.583584148	8.583587866	3.71747E-06	8.583580501	3.64733E-06
10.5	1.05	9.596754869	9.596758629	3.76026E-06	9.596750752	4.11729E-06
11	1.1	10.72752025	10.72752397	3.72569E-06	10.7275156	4.65029E-06
11.5	1.15	11.98877368	11.98877728	3.59702E-06	11.98876843	5.25447E-06
12	1.2	13.39476457	13.39476793	3.35484E-06	13.39475863	5.93896E-06
12.5	1.25	14.96124094	14.96124392	2.97671E-06	14.96123423	6.71388E-06
13	1.3	16.70560705	16.70560949	2.43669E-06	16.70559946	7.59072E-06
13.5	1.35	18.64709759	18.64709929	1.70492E-06	18.647089	8.58236E-06
14	1.4	20.80697016	20.8069709	7.47037E-07	20.80696045	9.70321E-06
14.5	1.45	23.20871805	23.20871758	4.76417E-07	23.20870708	1.09695E-05
15	1.5	25.87830538	25.87830337	2.01071E-06	25.87829299	1.23995E-05
15.5	1.55	28.84442692	28.84442301	3.9077E-06	28.84441291	1.40135E-05
16	1.6	32.1387953	32.13878907	6.2267E-06	32.13877946	1.58344E-05
16.5	1.65	35.79645838	35.79644935	9.0355E-06	35.79644049	1.7888E-05
17	1.7	39.85615007	39.85613766	1.24115E-05	39.85612987	2.0203E-05
17.5	1.75	44.36067794	44.3606615	1.64428E-05	44.36065513	2.28117E-05
18	1.8	49.35735167	49.35733044	2.12301E-05	49.35732592	2.57502E-05
18.5	1.85	54.89845654	54.89842965	2.68877E-05	54.89842748	2.90591E-05
19	1.9	61.04177674	61.04174319	3.35458E-05	61.04174396	3.27839E-05
19.5	1.95	67.85117366	67.85113231	4.13521E-05	67.85113668	3.69753E-05
20	2	75.39722505	75.39717458	5.04747E-05	75.39718336	4.16903E-05

Table 28: Numerical solution to problem 2 (z) and its error estimation

n	x	z	ABM4	Error	ABM5	Error
0	0	2				
0.5	0.05	2.215512754				
1	0.1	2.464208274				
1.5	0.15	2.749576423				
2	0.2	3.075474093	3.075473621	4.71858E-07		
2.5	0.25	3.446163812	3.446162758	1.05419E-06	3.446162323	1.48895E-06
3	0.3	3.866356401	3.86635486	1.5411E-06	3.866354668	1.73284E-06
3.5	0.35	4.341258122	4.341256151	1.97173E-06	4.34125615	1.97284E-06
4	0.4	4.876622785	4.876620148	2.63738E-06	4.876620551	2.23415E-06
4.5	0.45	5.478809333	5.478805923	3.41074E-06	5.478806769	2.5645E-06
5	0.5	6.154845485	6.154841218	4.26758E-06	6.154842565	2.9207E-06
5.5	0.55	6.912498072	6.912492783	5.28855E-06	6.912494758	3.31403E-06
6	0.6	7.760350768	7.760344284	6.48451E-06	7.760346993	3.77474E-06
6.5	0.65	8.707890003	8.707882139	7.86343E-06	8.707885713	4.28991E-06
7	0.7	9.765599901	9.765590438	9.46213E-06	9.765595035	4.86577E-06
7.5	0.75	10.94506721	10.9450559	1.13129E-05	10.94506169	5.52268E-06
8	0.8	12.25909727	12.25908383	1.34467E-05	12.25909101	6.26248E-06
8.5	0.85	13.72184218	13.72182627	1.59039E-05	13.72183508	7.0935E-06
9	0.9	15.34894239	15.34892366	1.87299E-05	15.34893436	8.0331E-06
9.5	0.95	17.15768333	17.15766135	2.19742E-05	17.15767424	9.09177E-06
10	1	19.1671683	19.1671426	2.56936E-05	19.16715801	1.02826E-05
10.5	1.05	21.39850974	21.39847979	2.99524E-05	21.39849811	1.16245E-05
11	1.1	23.8750405	23.87500568	3.48229E-05	23.87502736	1.31351E-05
11.5	1.15	26.62254736	26.62250698	4.03865E-05	26.62253253	1.4834E-05
12	1.2	29.66952914	29.66948241	4.67351E-05	29.6695124	1.67455E-05
12.5	1.25	33.04748188	33.04742791	5.39725E-05	33.04746299	1.88951E-05
13	1.3	36.79121411	36.79115189	6.22152E-05	36.79119279	2.13114E-05
13.5	1.35	40.93919517	40.93912358	7.15947E-05	40.93917115	2.40269E-05
14	1.4	45.53394031	45.53385805	8.2259E-05	45.53391324	2.70781E-05
14.5	1.45	50.62243611	50.62234173	9.43745E-05	50.6224056	3.05052E-05
15	1.5	56.25661077	56.25650264	0.000108129	56.25657642	3.43536E-05
15.5	1.55	62.49385384	62.49373011	0.000123732	62.49381517	3.86742E-05
16	1.6	69.39759059	69.39744917	0.000141421	69.39754707	4.35235E-05
16.5	1.65	77.03791676	77.0377553	0.000161463	77.0378678	4.8965E-05
17	1.7	85.49230014	85.49211599	0.000184155	85.49224507	5.50698E-05
17.5	1.75	94.84635588	94.84614604	0.000209835	94.84629396	6.1917E-05
18	1.8	105.1947033	105.1944645	0.00023888	105.1946337	6.95954E-05
18.5	1.85	116.6419131	116.6416414	0.000271713	116.6418349	7.8204E-05
19	1.9	129.3035535	129.3032447	0.00030881	129.3034656	8.78536E-05
19.5	1.95	143.3073473	143.3069966	0.000350705	143.3072486	9.86678E-05
20	2	158.7944501	158.7940521	0.000397996	158.7943393	0.000110785

Table 29: Numerical solution to problem 3 (x) and its error estimation when h=0.05

n	t	x	ABM4	Error	ABM5	Error
0	0	2				
0.5	0.05	2.410675336				
1	0.1	2.845477521				
1.5	0.15	3.308758202				
2	0.2	3.805154047	3.805153652	3.94753E-07		
2.5	0.25	4.339633152	4.339632666	4.86645E-07	4.339589912	4.32404E-05
3	0.3	4.917544765	4.917544124	6.40922E-07	4.917500508	4.4257E-05
3.5	0.35	5.544672819	5.544672063	7.55462E-07	5.544628275	4.45431E-05
4	0.4	6.227293821	6.227292598	1.22382E-06	6.227246841	4.69805E-05
4.5	0.45	6.972239674	6.972237865	1.80898E-06	6.972191228	4.8446E-05
5	0.5	7.786966044	7.78696349	2.55468E-06	7.786914594	5.14498E-05
5.5	0.55	8.679626988	8.679623514	3.47455E-06	8.679573076	5.39117E-05
6	0.6	9.659156556	9.659151933	4.62319E-06	9.659098248	5.83083E-05
6.5	0.65	10.73535821	10.73535225	5.96346E-06	10.73529655	6.16634E-05
7	0.7	11.91900294	11.91899538	7.55928E-06	11.91893554	6.73938E-05
7.5	0.75	13.22193705	13.22192764	9.41351E-06	13.22186498	7.20738E-05
8	0.8	14.65720076	14.65718918	1.15742E-05	14.65712146	7.92997E-05
8.5	0.85	16.23915865	16.23914459	1.40624E-05	16.23907324	8.54097E-05
9	0.9	17.98364351	17.98362657	1.6934E-05	17.98354898	9.45275E-05
9.5	0.95	19.90811471	19.90809448	2.02259E-05	19.90801237	0.000102342
10	1	22.03183301	22.03180901	2.40015E-05	22.03171937	0.000113646
10.5	1.05	24.37605331	24.37602499	2.83168E-05	24.37592974	0.000123568
11	1.1	26.96423734	26.96420409	3.3248E-05	26.96409983	0.000137507
11.5	1.15	29.82228852	29.82224965	3.88711E-05	29.82213856	0.000149957
12	1.2	32.97881119	32.97876591	4.52807E-05	32.97864409	0.000167098
12.5	1.25	36.46539688	36.46534431	5.25767E-05	36.46521424	0.000182641
13	1.3	40.31694053	40.31687965	6.08773E-05	40.31673688	0.000203648
13.5	1.35	44.57198966	44.57191935	7.03116E-05	44.57176668	0.000222983
14	1.4	49.27313025	49.27304922	8.10285E-05	49.27288156	0.000248695
14.5	1.45	54.46741289	54.4673197	9.31929E-05	54.46714021	0.000272674
15	1.5	60.2068237	60.20671671	0.000106993	60.20651959	0.000304116
15.5	1.55	66.54880464	66.548682	0.000122637	66.54847084	0.000333801
16	1.6	73.55682839	73.55668802	0.000140364	73.55645616	0.000372225
16.5	1.65	81.30103359	81.30087316	0.000160438	81.30062467	0.000408921
17	1.7	89.85892687	89.85874371	0.000183159	89.858471	0.000455869
17.5	1.75	99.31615849	99.31594963	0.000208862	99.31565731	0.000501181
18	1.8	109.7673796	109.7671417	0.000237925	109.7668211	0.000558539
18.5	1.85	121.3171896	121.3169188	0.00027077	121.3165751	0.000614443
19	1.9	134.0811827	134.0808748	0.000307874	134.0804982	0.000684521
19.5	1.95	148.1871054	148.1867556	0.000349769	148.186352	0.000753447
20	2	163.7761345	163.7757374	0.000397054	163.7752954	0.000839074

Table 30: Numerical solution to problem 3 (y) and its error estimation when h=0.05

n	t	y	ABM4	Error	ABM5	Error
0	0	2				
0.5	0.05	2.010008336				
1	0.1	2.040133511				
1.5	0.15	2.090677028				
2	0.2	2.162144744	2.162144528	2.15969E-07		
2.5	0.25	2.25525193	2.255251031	8.99656E-07	2.255263747	1.18162E-05
3	0.3	2.370930436	2.370929041	1.39506E-06	2.37093997	9.53335E-06
3.5	0.35	2.510338011	2.51033616	1.85152E-06	2.510344539	6.52743E-06
4	0.4	2.674869893	2.674867625	2.26799E-06	2.674874389	4.4962E-06
4.5	0.45	2.866172771	2.866170059	2.71186E-06	2.866174124	1.35353E-06
5	0.5	3.08616127	3.086158164	3.10572E-06	3.086160795	4.74432E-07
5.5	0.55	3.337037108	3.337033564	3.54389E-06	3.337033372	3.73613E-06
6	0.6	3.621311135	3.621307148	3.98647E-06	3.621305546	5.58858E-06
6.5	0.65	3.941828461	3.94182398	4.48095E-06	3.941819461	8.99973E-06
7	0.7	4.301796931	4.301791918	5.01248E-06	4.301786011	1.09202E-05
7.5	0.75	4.70481923	4.704813609	5.62122E-06	4.704804531	1.46997E-05
8	0.8	5.154928942	5.154922642	6.30044E-06	5.154912181	1.67613E-05
8.5	0.85	5.656630916	5.656623833	7.08287E-06	5.656609915	2.10004E-05
9	0.9	6.214946353	6.21493838	7.97292E-06	6.214923002	2.33504E-05
9.5	0.95	6.835463062	6.835454062	8.99979E-06	6.835434865	2.81969E-05
10	1	7.524391382	7.524381206	1.01757E-05	7.524360444	3.09382E-05
10.5	1.05	8.288626341	8.28861481	1.15304E-05	8.288589761	3.65795E-05
11	1.1	9.135816658	9.135803574	1.30837E-05	9.1357768	3.98579E-05
11.5	1.15	10.0744413	10.07442643	1.48683E-05	10.07439481	4.64856E-05
12	1.2	11.11389433	11.11387742	1.69121E-05	11.11384386	5.04778E-05
12.5	1.25	12.26457896	12.26455971	1.92532E-05	12.26452063	5.83325E-05
13	1.3	13.53801161	13.53798968	2.19289E-05	13.53794837	6.32398E-05
13.5	1.35	14.94693724	14.94691225	2.49851E-05	14.94686463	7.26085E-05
14	1.4	16.50545683	16.50542836	2.84701E-05	16.50537815	7.86824E-05
14.5	1.45	18.22916859	18.22913615	3.24408E-05	18.22907868	8.99051E-05
15	1.5	20.13532399	20.13528703	3.69589E-05	20.13522654	9.74514E-05
15.5	1.55	22.24300048	22.24295839	4.20955E-05	22.24288954	0.000110942
16	1.6	24.5732924	24.57324447	4.79289E-05	24.57317207	0.000120334
16.5	1.65	27.14952209	27.14946754	5.45485E-05	27.14938549	0.000136594
17	1.7	29.99747332	29.99741126	6.20534E-05	29.99732503	0.000148291
17.5	1.75	33.14564934	33.14557879	7.05559E-05	33.14548142	0.000167927
18	1.8	36.62555817	36.62547798	8.01815E-05	36.62537567	0.000182495
18.5	1.85	40.47202789	40.47193682	9.10715E-05	40.47182164	0.000206245
19	1.9	44.72355527	44.72345188	0.000103384	44.72333089	0.000224379
19.5	1.95	49.42269102	49.42257372	0.000117297	49.42243787	0.000253142
20	2	54.61646567	54.61633266	0.00013301	54.61618997	0.0002757

Table 31: Numerical solution to problem 4 (x) and its error estimation when h=0.05

n	t	x	ABM4	Error	ABM5	Error
0.0	1.0	1.0				
0.5	1.05	0.951229425				
1.0	1.1	0.904837418				
1.5	1.15	0.860707976				
2.0	1.2	0.818730753	0.818730806	5.31199E-08		
2.5	1.25	0.778800783	0.778800856	7.28939E-08	0.778800965	1.824E-07
3.0	1.3	0.740818221	0.740818271	5.08135E-08	0.740818291	6.98733E-08
3.5	1.35	0.70468809	0.704688154	6.41754E-08	0.704688147	5.77286E-08
4.0	1.4	0.670320046	0.670320175	1.28897E-07	0.670320079	3.34046E-08
4.5	1.45	0.637628152	0.637628352	2.00473E-07	0.63762799	1.61245E-07
5.0	1.5	0.60653066	0.60653095	2.90304E-07	0.606530288	3.72028E-07
5.5	1.55	0.57694981	0.576950261	4.50563E-07	0.576949276	5.34351E-07
6.0	1.6	0.548811636	0.548812341	7.0499E-07	0.548810732	9.04216E-07
6.5	1.65	0.522045777	0.522046849	1.07201E-06	0.522044211	1.56611E-06
7.0	1.7	0.496585304	0.49658692	1.6167E-06	0.496582943	2.36097E-06
7.5	1.75	0.472366553	0.472368995	2.44261E-06	0.472363066	3.48718E-06
8.0	1.8	0.449328964	0.449332643	3.67891E-06	0.449323576	5.38764E-06
8.5	1.85	0.427414932	0.427420449	5.51696E-06	0.427406734	8.19837E-06
9.0	1.9	0.40656966	0.406577918	8.25875E-06	0.406557511	1.21488E-05
9.5	1.95	0.386741023	0.386753377	1.23535E-05	0.386722876	1.81474E-05
10.0	2.0	0.367879441	0.367897901	1.84602E-05	0.36785213	2.73116E-05
10.5	2.05	0.349937749	0.349965313	2.75636E-05	0.349896971	4.07784E-05
11.0	2.1	0.332871084	0.332912222	4.11388E-05	0.33281039	6.0694E-05
11.5	2.15	0.316636769	0.316698152	6.13831E-05	0.316546093	9.0676E-05
12.0	2.2	0.301194212	0.301285781	9.15687E-05	0.301058736	0.000135476
12.5	2.25	0.286504797	0.286641374	0.000136577	0.286302836	0.00020196
13.0	2.3	0.272531793	0.272735481	0.000203688	0.272230636	0.000301157
13.5	2.35	0.259240261	0.259544016	0.000303755	0.258790793	0.000449468
14.0	2.4	0.246596964	0.247049927	0.000452963	0.245926447	0.000670517
14.5	2.45	0.234570288	0.235245729	0.000675441	0.233570388	0.0009999
15.0	2.5	0.22313016	0.224137331	0.001007171	0.221638666	0.001491495
15.5	2.55	0.212247974	0.213749777	0.001501803	0.210022985	0.002224989
16.0	2.6	0.201896518	0.20413585	0.002239332	0.198577886	0.003318632
16.5	2.65	0.192049909	0.195388948	0.003339039	0.18710015	0.004949758
17.0	2.7	0.182683524	0.187662301	0.004978777	0.175300328	0.007383196
17.5	2.75	0.173773943	0.181197681	0.007423738	0.162761151	0.011012792
18.0	2.8	0.165298888	0.176368229	0.011069341	0.148872818	0.01642607
18.5	2.85	0.157237166	0.17374235	0.016505183	0.132736557	0.024500609
19.0	2.9	0.149568619	0.174179016	0.024610396	0.11302377	0.036544849
19.5	2.95	0.142274072	0.178969895	0.036695823	0.087764815	0.054509257
20.0	3.0	0.135335283	0.190051304	0.05471602	0.054031185	0.081304098

Table 32: Numerical solution to problem 4 (y) and its error estimation when h=0.05

n	t	y	ABM4	Error	ABM5	Error
0.0	1.0	2.0				
0.5	1.05	1.902458849				
1.0	1.1	1.809674836				
1.5	1.15	1.721415953				
2.0	1.2	1.637461506	1.637461491	1.54268E-08		
2.5	1.25	1.557601566	1.557601492	7.399E-08	1.557602083	5.17098E-07
3.0	1.3	1.481636441	1.481636356	8.49416E-08	1.481637042	6.00843E-07
3.5	1.35	1.409376179	1.409376051	1.2877E-07	1.409376758	5.78691E-07
4.0	1.4	1.340640092	1.340639872	2.20269E-07	1.340640668	5.7628E-07
4.5	1.45	1.275256303	1.275255987	3.16323E-07	1.275257043	7.39907E-07
5.0	1.5	1.213061319	1.213060892	4.2782E-07	1.213062245	9.25412E-07
5.5	1.55	1.153899621	1.153899013	6.07491E-07	1.153900681	1.06029E-06
6.0	1.6	1.097623272	1.097622393	8.79026E-07	1.097624679	1.4068E-06
6.5	1.65	1.044091554	1.044090292	1.26116E-06	1.044093598	2.04413E-06
7.0	1.7	0.993170608	0.993168789	1.81905E-06	0.993173425	2.81742E-06
7.5	1.75	0.944733105	0.944730449	2.65643E-06	0.944737027	3.92159E-06
8.0	1.8	0.898657928	0.898654026	3.9026E-06	0.89866373	5.80224E-06
8.5	1.85	0.854829864	0.854824115	5.74905E-06	0.854838457	8.5931E-06
9.0	1.9	0.813139319	0.813130822	8.49788E-06	0.813151845	1.25254E-05
9.5	1.95	0.773482047	0.773469448	1.25984E-05	0.773500553	1.8506E-05
10.0	2.0	0.735758882	0.735740173	1.87098E-05	0.735786536	2.76537E-05
10.5	2.05	0.699875498	0.699847681	2.78168E-05	0.699916602	4.11042E-05
11.0	2.1	0.665742167	0.665700773	4.13947E-05	0.665803172	6.10048E-05
11.5	2.15	0.633273539	0.633211898	6.16408E-05	0.633364511	9.09721E-05
12.0	2.2	0.602388424	0.602296596	9.18275E-05	0.602524182	0.000135758
12.5	2.25	0.573009594	0.572872758	0.000136836	0.573211823	0.000202229
13.0	2.3	0.545063586	0.54485964	0.000203946	0.545365	0.000301414
13.5	2.35	0.518480521	0.518176508	0.000304013	0.518930233	0.000449712
14.0	2.4	0.493193928	0.492740709	0.000453219	0.493864678	0.00067075
14.5	2.45	0.469140576	0.468464881	0.000675695	0.470140698	0.001000122
15.0	2.5	0.44626032	0.445252898	0.001007423	0.447752026	0.001491706
15.5	2.55	0.424495948	0.422993895	0.001502052	0.426721139	0.002225191
16.0	2.6	0.403793036	0.401553457	0.002239579	0.407111861	0.003318825
16.5	2.65	0.384099817	0.380760535	0.003339282	0.389049759	0.004949941
17.0	2.7	0.365367048	0.360388032	0.004979016	0.372750419	0.00738337
17.5	2.75	0.347547887	0.340123914	0.007423973	0.358560845	0.011012958
18.0	2.8	0.330597776	0.319528204	0.011069572	0.347024005	0.016426229
18.5	2.85	0.314474333	0.297968922	0.016505411	0.338975093	0.02450076
19.0	2.9	0.299137238	0.274526619	0.02461062	0.335682232	0.036544993
19.5	2.95	0.284548143	0.247852101	0.036696042	0.339057537	0.054509394
20.0	3.0	0.270670566	0.215954332	0.054716235	0.351974796	0.081304229

4. CONCLUSION

Linear multistep methods are used for solving ordinary differential equations (ODEs) with initial value problems (IVPs). To verify their convergence, two examples of second-order differential equations and two examples of coupled differential equations were considered using two different step sizes (0.1 and 0.05). The numerical solutions obtained by both methods showed good agreement with the exact solutions. Comparing the results of starting the solution process with Euler's method and the fourth-order Runge-Kutta (RK4) method, it was observed that using RK4 to initiate the Adams-Bashforth-Moulton method yielded more accurate results. The approximate solutions converged faster to the exact solutions when starting with RK4 compared to Euler's method. Therefore, it can be concluded that starting the solution process with RK4 in the Adams-Bashforth-Moulton method is a powerful and more efficient approach for finding numerical solutions to IVPs.

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