International Journal of Mathematics, Statistics, and Computer Science
Vol. 3, pp. 294-312
ISSN: 2704-1077 eISSN 2704-1069, DOI: https://doi.org/10.59543/ijmscs.v3i.10625

Fundamentals of Fermatean Neutrosophic Soft Set with Application in Decision Making Problem

Muhammad Saeed[1*], Imrana Shafique[1], Hatıra Günerhan[2,3]
[1]Department of Mathematics, University of Management and Technology, Lahore, Pakistan
[2]Department of Mathematics, Faculty of Education, Kafkas University, Kars, Turkey
[3]MEU Research Unit, Middle East University, Amman, Jordan
*Corresponding author: muhammad.saeed@umt.edu.pk

Received 03.09.2024, Revised 09.11.2024, Accepted 02.01.2025

ABSTRACT: The classical set theory based on crisp sets is not able to deal with uncertainties which is a common feature of various real-world problems. This problem is solved using modified forms of sets such as fuzzy sets, Intuitionistic fuzzy sets, neutrosophic sets, soft sets and hypersoft sets and others along with their hybrids. In this paper, a modified hybrid of soft set named Fermatean Neutrosophic Soft set ($FrNSS$) is established. Basic entities of set theory including subsets, null set, universal set along with different operators are defined. With respect to these operators, the algebraic structures as monoid, semigroup and semiring are defined. Also, fermatean neutrosophic soft topological space and the cartesian product of fermatean neutrosophic soft sets and fermatean neutrosophic soft relation are defined to establish an application of this hybrid structure to decision-making problems.

1. INTRODUCTION

Around the 1870's, set theory was introduced as a result of Cantor and Dedekind's efforts, [1] which proved its worth having various real-world applications. The classical set theory based on crisp sets only deals with absolute associateship that is, whether a member is contained in a set or not. This limitation of associateship motivated Zadeh to introduce fuzzy sets that deal with partial associateship [2]. Fuzzy sets were introduced in 1965, generalized by Pawlak as rough sets in 1982 [3] and by Molodstov as soft sets in 1999 [4]. These generalizations proved their worth in dealing with the uncertainties in various real-world problems in almost every field such as engineering, economics, social sciences, environmental sciences and medical sciences [5–8].

Fuzzy soft set [7], intuitionistic fuzzy set [9], intuitionistic fuzzy soft set [10], hesitant fuzzy set [11], hesitant fuzzy soft set [12], picture fuzzy set [13], picture fuzzy soft set [14], hypersoft set [15], neutrosophic soft set [16] and neutrosophic hypersoft set [17] are few variants based on generalization of truthiness (associateship), falsity (non-associateship) and hesitancy (indeterminacy). We have enlightened some scholarly activities related to soft sets, neutrosophic sets and fermatean sets. In 2003, Maji proposed the fundamentals of soft sets including basic entities and operators [18] and in 2009 Ali et al. established the modified operators [19]. Later, the soft set theory evolved as Çağman and Enginoğlu proposed the soft matrix [20], Babitha and Sunil defined relations and functions on soft sets along with their properties [21, 22] and Yang and Guo defined closure and kernel of soft relations and soft mappings [23]. More contributions towards soft set theory were made by different mathematicians [24–27]. While soft set theory was developing by above mentioned findings, mathematicians tried to connect it with algebraic structures as Aktaş and Çağman established soft groups [28], Acer defined soft rings [5] and Aslam and Qurashi connected sub algebraic structures related to soft groups [29].

Inspired by philosophical logics (relative and absolute truthiness and falsity) and various real-world scenarios such as game results (win, loss, tie), voting outcomes (in favour of, opposite, blank vote), numbers (positive, negative, neutral), answers to a straight question (yes, not applicable, no), control theory and decision-making (making a decision, hesitating, accepting, rejecting, pending). Smarandache introduced a tri-component set named as neutrosophic sets (knowledge of neutral wisdom) dealing with three components: associateship, non-associateship and indeterminacy [30, 31]. Neutrosophic set theory evolved as Wang et al. proposed single and interval valued neutrosophic sets [32, 33] and Salama and Alblowi presented Neutrosophic

Table 1. Alreday existing structures

Set	**associateship value θ**	**Indeterminacy value ϕ**	**Non-associateship value ψ**	**condition**
	m	i	n	
Crisp Set	0 or 1	0	0	
Fuzzy Set	in $[0, 1]$	0	0	
Intuitionistic Fuzzy Set	in $[0, 1]$	0	in $[0, 1]$	$0 \leq \theta + \psi \leq 1$
Pythagorean Fuzzy Set	in $[0, 1]$	0	in $[0, 1]$	$0 \leq \theta^2 + \psi^2 \leq 1$
Fermatean Fuzzy Set	in $[0, 1]$	0	in $[0, 1]$	$0 \leq \theta^3 + \psi^3 \leq 1$
Neutrosophic Set	in $[0, 1]$	in $[0, 1]$	in $[0, 1]$	$0 \leq \theta + \phi + \psi \leq 3$
Intuitionistic Neutrosophic Set	in $[0, 1]$	in $[0, 1]$	in $[0, 1]$	$0 \leq \theta + \psi \leq 1$ $0 \leq \theta + \phi + \psi \leq 2$
Pythagorean Neutrosophic Set	in $[0, 1]$	in $[0, 1]$	in $[0, 1]$	$0 \leq \theta^2 + \psi^2 \leq 1$ $0 \leq \theta^2 + \phi^2 + \psi^2 \leq 2$
Fermatean Neutrosophic Set	in $[0, 1]$	in $[0, 1]$	in $[0, 1]$	$0 \leq \theta^3 + \psi^3 \leq 1$ $0 \leq \theta^3 + \phi^3 + \psi^3 \leq 2$

topological spaces [34]. Georgiou introduced soft topological spaces [35] and Bera and Mahapatra introduced neutrosophic soft topological spaces [36]. Various mathematical entities were established relative to neutrosophic set as measure and integral [37], lattices [38], vector spaces [39], continuous function [40], entropy [41], group and subgroup [42], soft ring and soft field [43]. Mathematicians also discussed various applications of neutrosophic techniques including image processing [44], medical diagnosis [45, 46] and multi-criteria decision-making [47–49] using similarity measures, neutrosophic logic and hypersoft graphs [50]. Senapati and Yager introduced fermatean fuzzy set [51] to deal with the limitations of associateship and non-associateship in intuitionistic fuzzy and pythagorean fuzzy sets. It opened a new horizon for researchers as Broumi et al. applied complex fermatean neutrosophic graphs to decision-making [52], Bilgin et al. introduced fermatean neutrosophic topological spaces [53] and Salsabeela and John discussed TOPSIS techniques on fermatean fuzzy soft sets [54].

This paper presents the fundamentals of a hybrid structure fermatean Neutrosophic Soft set ($FrNSS$) that allows more flexible choices for associateship, non-associateship and indeterminacy. We have established its definition, basic set theoretic entities as subset, null set, universal set, different operators and basic algebraic structures relative to these operators. We have also defined fermatean neutrosophic soft topological space, cartesian product, relations on $FrNSS$ and discussed its approach to decision-making problem.

2. STRUCTURAL COMPARISON

In this section, we have presented fermatean neutrosophic set as a generalization of some basic hybrids. Table 1 shows how different values of associateship, indeterminacy and non-associateship correspond to other already existing hybrid structures and some basic sets. It is to be noted that fermatean neutrosophic set is not a special case of q-rung orthopair fuzzy set with q=3, as in q-rung orthopair fuzzy set only associateship and non-associateship (dependent) are discussed while in case of fermatean neutrosophic set, associateship, non-associateship (dependent) and indeterminacy (independent) are discussed. Pythagorean neutrosophic set is generalization of intuitionistic neutrosophic set and fermatean neutrosophic set is a generalization of both pythagorean neutrosophic and Intuitionistic neutrosophic sets.

3. MOTIVATION

In this section a few real-world scenerios that neutrosophic set deals with, are presented. The hybrid structure $FrNSS$ proves its worth being able to deal with more options for associateship, non-associateship and indeterminacy as compared to itutionistic and pythagorean neutrosophic sets.

3.1. General Example

Problem: During the journey from place A to place B, a truck is loaded with various items, including tables of three different sizes (large (table 1), medium (table 2) and small (table 3)), a sofa set with three different sizes (three-seater (sofa 1), two-seater (sofa 2) and one-seater (sofa 3)), a cupboard and two boxes. Table 2 is positioned on top of table 1 and table 3 is placed underneath table 1. Sofa 3 is placed on top of sofa 1 while sofa 2 is inclined against sofa 1 forming a slopy surface. Now let's express the volume covered by each item on the truck throughout the entire journey.
Solution: In this particular problem, the coverage of volume by each item is not absolute. For example table 2 does not cover the volume of the truck even though it is present in the truck. Sofa 2 formimg an inclined plane with sofa 1 covers approximately 50% to 60% of the area (associateship) while the remaining 50% to 40% does not cover the volume of the truck (non-associateship). Moreover, the movement of the truck introduces frequent changes in these values of associateship and non-associateship. This problem cannot be effectively addressed using crisp sets. To analyse and discuss this problem, we require a neutrosophic soft structure that accounts for the dependencies between associateship and non-associateship. The most suitable hybrid structure for this problem is the $FrNSS$ which allows a wide range of possible values for associateship, indeterminacy, and non-membership. e.g if associateship is 0.9 (90%) and non-associateship is 0.5 (50%) then $0.9 + 0.5 > 1$, so intuitionistic neutrosophic soft set does not support it. Also $0.9^2 + 0.5^2 > 1$ so pythagorean neutrosophic soft set does not support it but $0.9^3 + 0.5^3 < 1$ so fermatean neutrosophic soft set will support it.

3.2. Neutrosophy in Quantum Mechanics

Both wave and particle are characteristics of photons [55]. Independently, neither the photon's particle character nor its wave nature can account for the phenomenon of light. The particle nature of photons elucidates their straight-line motion, while their wave nature accounts for phenomenon like reflection.
The neutrosophic nature of sets finds its most valuable application in describing the quantum state of photons, which exists in a superposition, manifesting as two distinct states. This complex situation can be effectively represented using the fermatean neutrosophic set which encompasses a wide range of potential values for associateship, non-associateship, and indeterminacy.

3.3. Neutrosophy in particle physics

Supersymmetry (SUSY) is a theory that proposes the existence of a connection between bosons (particles with zero or integral spin) and fermions (particles with half-integer spin). It postulates that these particles can be organised into the same doublet and introduces a supercharge operator, denoted as Q, which can transform fermions into bosons and vice versa.
In order to show an unbroken symmetry, fermions and bosons can be conceptualised as Neutrosophic states possessing opposing properties such as spin and statistics. The SUSY doublet serves as a neutral term that accommodates both types of particles within the framework of supersymmetry. [56]

3.4. Neutrosophy and accelerated expansion of universe

The Nobel Prize in Physics was awarded in 2011 for the groundbreaking discovery of the universe's accelerated expansion. This phenomenon can be effectively expressed using neutrosophy which encompasses three states, expansion, contraction, and a stable state characterised by neither expansion nor contraction. Neutrosophy provides a suitable framework to capture the complex dynamics of the universe's evolution [57].
Researchers have discussed many real-world applications of neutrosphy, a few of which are mentioned above. An important point to notice is that the dual nature of photons is interdependent, so neutrosophic structure cannot deal with it and we need to develop a hybrid neutrosophic structure to discuss such scientific scenarios. To have an extended domain, we have developed fermatean neutrosophic soft set which is an extension of intuitionistic and pythagorean neutrosophic structures.

4. PRELIMINARIES

In order to comprehend the paper's main findings, some basic definitions, mainly following [53], [58] and [36] are presented in this section. Let's define few notations that we have used for this paper. $\mathbb{D}, \mathbb{P}(\mathbb{D})$, $\mathbb{P}(\mathbb{D})_{FrN}$ and $\mathbb{P}(\mathbb{D})_N$ are used to represent the domain of discourse, collection of all the classical subsets of $\mathbb{D}$, fermatean neutrosophic subsets of $\mathbb{D}$ and neutrosophic subsets of $\mathbb{D}$, respectively. P_1 and P_2 are used to represent the subsets of set of parameters P. $\theta_X, \phi_X, \psi_X : \mathbb{D} \to [0,1]$ where $\theta_X(\tilde{s}), \phi_X(\tilde{s})$ and $\psi_X(\tilde{s})$ are

representing associateship, indeterminacy and non-associateship levels of $\tilde{s} \in \mathbb{D}$ relative to the set X.
The collection of possible values for fermatean neutrosophy associateship and non-associateship levels is a super set of the collections of pythagorean as well as intuitionistic associateship and non-associateship levels.

4.1. Fermatean Fuzzy Set

$\xi_{Fr} = \{\langle \tilde{s}, (\theta_\xi(\tilde{s}), \psi_\xi(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}, 0 \leq \theta_\xi^3(\tilde{s}) + \psi_\xi^3(\tilde{s}) \leq 1\}$ is representing a fermatean fuzzy set over the domain of discourse $\mathbb{D}$.

4.2. Neutrosophic Set

$\xi_N = \{\langle \tilde{s}, (\theta_\xi(\tilde{s}), \phi_\xi(\tilde{s}), \psi_\xi(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}, 0 \leq \theta_\xi(\tilde{s}) + \phi_\xi(\tilde{s}) + \psi_\xi(\tilde{s}) \leq 3\}$ is representing a neutrosophic set over the domain of discourse $\mathbb{D}$.

4.3. Soft Set

The pair $(f^*, P_1) = \{(\hat{\epsilon}, f^*(\hat{\epsilon})) : \hat{\epsilon} \in P_1, f^* : P_1 \rightarrow \mathbb{P}(\mathbb{D})\}$ is representing a soft set.

4.4. Fermatean Neutrosophic Set

$\xi_{FrN} = \{\langle \tilde{s}, (\theta_\xi(\tilde{s}), \phi_\xi(\tilde{s}), \psi_\xi(\tilde{s}))\rangle, 0 \leq \theta_\xi^3(\tilde{s}) + \psi_\xi^3(\tilde{s}) \leq 1, 0 \leq \theta_\xi^3(\tilde{s}) + \phi_\xi^3(\tilde{s}) + \psi_\xi^3(\tilde{s}) \leq 2 : \tilde{s} \in \mathbb{D}\}$ is representing a fermatean neutrosophic set over the domain of discourse $\mathbb{D}$.

4.5. Neutrosophic Soft Set

The pair $(f^*, P_1) = \{(\hat{\epsilon}, f^*(\hat{\epsilon})) : \hat{\epsilon} \in P_1, f^* : P_1 \rightarrow \mathbb{P}(\mathbb{D})_N\}$ is representing a neutrosophic soft set. More precisely,
$\xi_{NS,P_1} =$
$\{(\hat{\epsilon}, \{\langle \tilde{s}, (\theta_{P_1,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}), \psi_{P_1,\hat{\epsilon}}(\tilde{s}))\rangle, 0 \leq \theta_{P_1,\hat{\epsilon}}(\tilde{s}) + \phi_{P_1,\hat{\epsilon}}(\tilde{s}) + \psi_{P_1,\hat{\epsilon}}(\tilde{s}) \leq 3 : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_1\}$

4.6. Neutrosophic Soft Subset

A neutrosophic soft set $\xi_{\mathsf{NS},P_1} =$
$\{(\hat{\epsilon}, \{\langle \tilde{s}, (\theta_{P_1,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}), \psi_{P_1,\hat{\epsilon}}(\tilde{s}))\rangle, 0 \leq \theta_{P_1,\hat{\epsilon}}(\tilde{s}) + \phi_{P_1,\hat{\epsilon}}(\tilde{s}) + \psi_{P_1,\hat{\epsilon}}(\tilde{s}) \leq 3 : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_1\}$ is considered to be a neutrosophic soft subset of $\xi_{\mathsf{NS},P_2} =$
$\{(\hat{\epsilon}, \{\langle \tilde{s}, (\theta_{P_2,\hat{\epsilon}}(\tilde{s}), \phi_{P_2,\hat{\epsilon}}(\tilde{s}), \psi_{P_2,\hat{\epsilon}}(\tilde{s}))\rangle, 0 \leq \theta_{P_2,\hat{\epsilon}}(\tilde{s}) + \phi_{P_2,\hat{\epsilon}}(\tilde{s}) + \psi_{P_2,\hat{\epsilon}}(\tilde{s}) \leq 3 : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_2\}$ if (i) $P_1 \subseteq P_2$, (ii) $\theta_{P_1,\hat{\epsilon}}(\tilde{s}) \leq \theta_{P_2,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}) \leq \phi_{P_2,\hat{\epsilon}}(\tilde{s})$ and $\psi_{P_1,\hat{\epsilon}}(\tilde{s}) \geq \psi_{P_1,\hat{\epsilon}}(\tilde{s})$, for all $\tilde{s} \in \mathbb{D}, \hat{\epsilon} \in P_1$.

4.7. Neutrosophic Soft Twisted Subset

A neutrosophic soft set $\xi_{\mathsf{NS},P_1} =$
$\{(\hat{\epsilon}, \{\langle \tilde{s}, (\theta_{P_1,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}), \psi_{P_1,\hat{\epsilon}}(\tilde{s}))\rangle, 0 \leq \theta_{P_1,\hat{\epsilon}}(\tilde{s}) + \phi_{P_1,\hat{\epsilon}}(\tilde{s}) + \psi_{P_1,\hat{\epsilon}}(\tilde{s}) \leq 3 : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_1\}$ is considered to be a neutrosophic soft twisted subset of $\xi_{\mathsf{NS},P_2} =$
$\{(\hat{\epsilon}, \{\langle \tilde{s}, (\theta_{P_2,\hat{\epsilon}}(\tilde{s}), \phi_{P_2,\hat{\epsilon}}(\tilde{s}), \psi_{P_2,\hat{\epsilon}}(\tilde{s}))\rangle, 0 \leq \theta_{P_2,\hat{\epsilon}}(\tilde{s}) + \phi_{P_2,\hat{\epsilon}}(\tilde{s}) + \psi_{P_2,\hat{\epsilon}}(\tilde{s}) \leq 3 : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_2\}$ if (i) $P_1 \subseteq P_2$, (ii) $\theta_{P_1,\hat{\epsilon}}(\tilde{s}) \geq \theta_{P_2,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}) \geq \phi_{P_2,\hat{\epsilon}}(\tilde{s})$ and $\psi_{P_1,\hat{\epsilon}}(\tilde{s}) \leq \psi_{P_1,\hat{\epsilon}}(\tilde{s})$, for all $\tilde{s} \in \mathbb{D}, \hat{\epsilon} \in P_1$.

4.8. Relative Null and Relative Whole Neutrosophic Soft Set

A neutrosophic soft set, $\xi_{NS,P_1} = \{(\hat{\epsilon}, \{\langle \tilde{s}, (0,0,1)\rangle : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_1\}$ is named as relative null neutrosophic soft set and $\xi_{NS,P_1} = \{(\hat{\epsilon}, \{\langle \tilde{s}, (1,1,0)\rangle : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_1\}$ is named as relative whole neutrosophic soft set.

4.9. Operations on Neutrosophic Soft Sets

Following are few operations defined on neutrosophic soft sets,
(i) Complement: $\xi^c_{\mathsf{NS},P_1} = \{(\hat{\epsilon}, \{\langle \tilde{s}, (\psi_{P_1,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}), \theta_{P_1,\hat{\epsilon}}(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_1\}$.
(ii) Restricted unoin:
$\xi_{\mathsf{NS},P_1} \cup_R \xi_{\mathsf{NS},P_2} = \xi_{\mathsf{NS},P_3} =$
$\{(\hat{\epsilon}, \{\langle \tilde{s}, (\max\{\theta_{P_1,\hat{\epsilon}}(\tilde{s}), \theta_{P_2,\hat{\epsilon}}(\tilde{s})\}, \max\{\phi_{P_1,\hat{\epsilon}}(\tilde{s}), \phi_{P_2,\hat{\epsilon}}(\tilde{s})\}, \min\{\psi_{P_1,\hat{\epsilon}}(\tilde{s}), \psi_{P_2,\hat{\epsilon}}(\tilde{s})\})\rangle : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_3\}, P_3 = P_1 \cap P_2$.
(iii) Restricted intersection:
$\xi_{\mathsf{NS},P_1} \cap_R \xi_{\mathsf{NS},P_2} = \xi_{\mathsf{NS},P_3} =$
$\{(\hat{\epsilon}, \{\langle \tilde{s}, (\min\{\theta_{P_1,\hat{\epsilon}}(\tilde{s}), \theta_{P_2,\hat{\epsilon}}(\tilde{s})\}, \min\{\phi_{P_1,\hat{\epsilon}}(\tilde{s}), \phi_{P_2,\hat{\epsilon}}(\tilde{s})\}, \max\{\psi_{P_1,\hat{\epsilon}}(\tilde{s}), \psi_{P_2,\hat{\epsilon}}(\tilde{s})\})\rangle : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_3\}, P_3 = P_1 \cap P_2$.
(iv) Extended unoin and intersection: $\xi_{\mathsf{NS},P_3} =$
$\{(\hat{\epsilon}, \{\langle \tilde{s}, (\theta_{P_3,\hat{\epsilon}}(\tilde{s}), \phi_{P_3,\hat{\epsilon}}(\tilde{s}), \psi_{P_3,\hat{\epsilon}}(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}\}) : \hat{\epsilon} \in P_3\}, P_3 = P_1 \cup P_2$, where associateship, indeterminacy and nonassociateship values are mentioned in table 2.

Table 2. associateship, indeterminacy and nonassociateship function values for extended union and intersection

partition of $P_3 = P_1 \cup P_2$	associateship value	Indeterminacy value	Non-associateship value
	$P_1 \cup_E P_2$		
$\hat{\epsilon} \in P_1 \setminus P_2$	$\theta_{P_1,\hat{\epsilon}}$	$\phi_{P_1,\hat{\epsilon}}$	$\psi_{P_1,\hat{\epsilon}}$
$\hat{\epsilon} \in P_2 \setminus P_1$	$\theta_{P_2,\hat{\epsilon}}$	$\phi_{P_2,\hat{\epsilon}}$	$\psi_{P_2,\hat{\epsilon}}$
$\hat{\epsilon} \in P_1 \cap P_2$	$\max\{\theta_{P_1,\hat{\epsilon}}, \theta_{P_2,\hat{\epsilon}}\}$	$\max\{\phi_{P_1,\hat{\epsilon}}, \phi_{P_2,\hat{\epsilon}}\}$	$\min\{\psi_{P_1,\hat{\epsilon}}, \psi_{P_2,\hat{\epsilon}}\}$
	$P_1 \cap_E P_2$		
$\hat{\epsilon} \in P_1 \setminus P_2$	$\theta_{P_1,\hat{\epsilon}}$	$\phi_{P_1,\hat{\epsilon}}$	$\psi_{P_1,\hat{\epsilon}}$
$\hat{\epsilon} \in P_2 \setminus P_1$	$\theta_{P_2,\hat{\epsilon}}$	$\phi_{P_2,\hat{\epsilon}}$	$\psi_{P_2,\hat{\epsilon}}$
$\hat{\epsilon} \in P_1 \cap P_2$	$\min\{\theta_{P_1,\hat{\epsilon}}, \theta_{P_2,\hat{\epsilon}}\}$	$\min\{\phi_{P_1,\hat{\epsilon}}, \phi_{P_2,\hat{\epsilon}}\}$	$\max\{\psi_{P_1,\hat{\epsilon}}, \psi_{P_2,\hat{\epsilon}}\}$

Table 3. Tabular form of FrNSS $\mathfrak{X}_{FrNS,P_1}$

$\mathfrak{X}_{FrNS,P_1}$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$
$\hat{\epsilon}_1$	(0.8, 0.4, 0.1)	(0.9, 0.7, 0.3)	(0.1, 0.2, 0.3)
$\hat{\epsilon}_2$	(0.6, 0.2, 0.4)	(0.8, 0.7, 0.3)	(0.1, 0.5, 0.7)

4.10. Neutrosophic Soft Topological space

Let $NSS(\mathbb{D}, P_1)$ be a collection of all neutrosophic soft sets over $\mathbb{D}$ with respect to the set of parameters P_1 and τ_{nsp_1} be a subset of $NSS(\mathbb{D}, P_1)$. τ_{nsp_1} is named as neutrosophic soft topology on $(\mathbb{D}, P_1)$ if (i) relative null and relative whole neutrosophic soft sets belong to τ_{nsp_1}, (ii) the intersection of finite number of neutrosophic soft sets in τ_{nsp_1} also belongs to τ_{nsp_1}, (iii) the union of any number of neutrosophic soft sets in τ_{nsp_1} also belongs to τ_{nsp_1}.
The triplet $(\mathbb{D}, P_1, \tau_{nsp_1})$ is named as neutrosophic soft topological space.

4.11. Neutrosophic Soft Cartesian Product

The cartesian product $\xi_{NS,P_1} \times \xi_{NS,P_2}$ is a neutrosophic soft set defined by,
$\xi_{NS,P_1} \times \xi_{NS,P_2} = \xi_{NS,P_3}$
$= \{(\hat{\epsilon}, \hat{\epsilon'}), \langle \tilde{s}, (\min\{\theta_{P_1,\hat{\epsilon}}(\tilde{s}), \theta_{P_2,\hat{\epsilon'}}(\tilde{s})\}, \min\{\phi_{P_1,\hat{\epsilon}}(\tilde{s}), \phi_{P_2,\hat{\epsilon'}}(\tilde{s})\}, \max\{\psi_{P_1,\hat{\epsilon}}(\tilde{s}), \psi_{P_2,\hat{\epsilon'}}(\tilde{s})\}) : \tilde{s} \in \mathbb{D}\rangle : (\hat{\epsilon}, \hat{\epsilon'}) \in P_1 \times P_2\}$

5. FERMATEAN NEUTROSOPHIC SOFT SET

In this section, a noval hybrid is established, possessing the properties of fermatean, neutrosophic and soft sets.

5.1. Definition

For the domain of discourse $\mathbb{D}$ and the collection of parameters P, define a mapping $f^* : P_1 \rightarrow \mathbb{P}(\mathbb{D})_{FrN}$, where P_1 is a non empty subset of P and $\mathbb{P}(\mathbb{D})_{FrN}$ is collection of all fermatean neutrosophic subsets of $\mathbb{D}$. The fermatean neutrosophic soft set ($FrNSS$) is defined as,
$\mathfrak{X}_{FrNS,P_1} = (f^*, P_1) = \{(\hat{\epsilon}, \langle \tilde{s}, (\theta_{P_1,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}), \psi_{P_1,\hat{\epsilon}}(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P_1\}$ where $\theta_{P_1,\hat{\epsilon}}, \phi_{P_1,\hat{\epsilon}}, \psi_{P_1,\hat{\epsilon}} : \mathbb{D} \rightarrow [0,1]$ such that for all $\hat{\epsilon} \in \mathbb{D}$ and $\hat{\epsilon} \in P_1, 0 \leq \theta^3_{P_1,\hat{\epsilon}}(\tilde{s}) + \psi^3_{P_1,\hat{\epsilon}}(\tilde{s}) \leq 1$ and $0 \leq \theta^3_{P_1,\hat{\epsilon}}(\tilde{s}) + \phi^3_{P_1,\hat{\epsilon}}(\tilde{s}) + \psi^3_{P_1,\hat{\epsilon}}(\tilde{s}) \leq 2$.

5.1.1. Example

Let $\mathbb{D} = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3\}, P = \{\hat{\epsilon}_1, \hat{\epsilon}_2, ..., \hat{\epsilon}_5\}$ and $P_1 = \{\hat{\epsilon}_1, \hat{\epsilon}_2\}$. Following is an example of $FrNSS$,
$\mathfrak{X}_{FrNS,P_1} = \{(\hat{\epsilon}_1, \langle \tilde{s}_1, (0.8, 0.4, 0.1)\rangle, \langle \tilde{s}_2, (0.9, 0.7, 0.3)\rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.3)\rangle),$
$(\hat{\epsilon}_2, \langle \tilde{s}_1, (0.6, 0.2, 0.4)\rangle, \langle \tilde{s}_2, (0.8, 0.7, 0.3)\rangle, \langle \tilde{s}_3, (0.1, 0.5, 0.7)\rangle)\}$. Table 3 is representing the tabular form of $FrNSS$.

5.2. Fermatean Neutrosophic Soft Subset

Over the same domain of discourse, A $FrNSS$, $\mathfrak{X}_{FrNS,P_1}$ is considered as a $FrNS$ subset of $\mathfrak{X}_{FrNS,P_2}$ if (i) $P_1 \subseteq P_2$, (ii) for all $\hat{\epsilon} \in P_1$ and $\tilde{s} \in \mathbb{D}, \theta_{P_1,\hat{\epsilon}}(\tilde{s}) \leq \theta_{P_2,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}) \leq \phi_{P_2,\hat{\epsilon}}(\tilde{s})$ and $\psi_{P_1,\hat{\epsilon}}(\tilde{s}) \geq \psi_{P_2,\hat{\epsilon}}(\tilde{s})$.

Remark: Its a clear observation that the definition of classical subset does not hold here as $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_2}$ does not imply that all the points of $\mathfrak{X}_{FrNS,P_1}$ are present in $\mathfrak{X}_{FrNS,P_2}$.

5.2.1. Example

Consider the $FrNSS$ $\mathfrak{X}_{FrNS,P_1}$ considered in example 5.1.1. and let $\mathfrak{X}_{FrNS,P_2}$ be another $FrNSS$ over the same domain of discourse given as $\mathfrak{X}_{FrNS,P_2} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.9, 0.5, 0.1)\rangle, \langle\tilde{s}_2, (0.9, 0.8, 0.1)\rangle, \langle\tilde{s}_3, (0.4, 0.5, 0.2)\rangle),$
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.7, 0.3, 0.2)\rangle, \langle\tilde{s}_2, (0.8, 0.7, 0.2)\rangle, \langle\tilde{s}_3, (0.1, 0.7, 0.4)\rangle),$
$(\hat{\epsilon}_4, \langle\tilde{s}_1, (0.5, 0.2, 0.6)\rangle, \langle\tilde{s}_2, (0.7, 0.3, 0.5) >\rangle, \langle\tilde{s}_3, (0.4, 0.2, 0.6)\rangle)\}$, where $P_2 = \{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_4\}$. Clearly, $\mathfrak{X}_{FrNS,P_1}$ is $FrNS$ subset of $\mathfrak{X}_{FrNS,P_2}$.

5.3. Fermatean Neutrosophic Soft Twisted Subset

Over the same domain of discourse, $\mathfrak{X}_{FrNS,P_1}$ is considered to be a $FrNS$ twisted subset of $\mathfrak{X}_{FrNS,P_2}$ if (i) $P_1 \subseteq P_2$, (ii) for all $\hat{\epsilon} \in P_1$ and $\tilde{s} \in \mathbb{D}, \theta_{P_1,\hat{\epsilon}}(\tilde{s}) \geq \theta_{P_2,\hat{\epsilon}}(\tilde{s}), \phi_{P_1,\hat{\epsilon}}(\tilde{s}) \geq \phi_{P_2,\hat{\epsilon}}(\tilde{s})$ and $\psi_{P_1,\hat{\epsilon}}(\tilde{s}) \leq \psi_{P_2,\hat{\epsilon}}(\tilde{s})$.

5.3.1. Example

Consider the $FrNSS$, $\mathfrak{X}_{FrNS,P_1}$ in example 5.1.1. and let $\mathfrak{X}_{FrNS,P_3} =$
$\{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.5, 0.2, 0.1)\rangle, \langle\tilde{s}_2, (0.7, 0.5, 0.6)\rangle, \langle\tilde{s}_3, (0.1, 0.1, 0.5)\rangle),$
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.4, 0.1, 0.7)\rangle, \langle\tilde{s}_2, (0.5, 0.5, 0.5)\rangle, \langle\tilde{s}_3, (0.1, 0.2, 0.9)\rangle),$
$(\hat{\epsilon}_3, \langle\tilde{s}_1, (0.4, 0.2, 0.3)\rangle, \langle\tilde{s}_2, (0.6, 0.2, 0.1)\rangle, \langle\tilde{s}_3, (0.5, 0.2, 0.5)\rangle)\}$, where $P_3 = \{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_3\}$. Clearly, $\mathfrak{X}_{FrNS,P_1}$ is $FrNS$ twisted subset of $\mathfrak{X}_{FrNS,P_3}$.

5.4. Fermatean Neutrosophic Soft Equal Set

Over the same domain of discourse, two $FrNSS$s $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ are considered to be $FrNS$ equal, if either $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_2}$ and $\mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1}$ or $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\tilde{\subseteq}} \mathfrak{X}_{FrNS,P_2}$ and $\mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\tilde{\subseteq}} \mathfrak{X}_{FrNS,P_1}$.

5.5. Relative Null Fermatean Neutrosophic Soft set

A $FrNSS$ $\mathfrak{X}_{FrNS,P_1}$ is considered as relative null $FrNSS$ ($\emptyset_{FrNS,P_1}$) if for all $\hat{\epsilon} \in P_1, \tilde{s} \in \mathbb{D}, \theta_{P_1,\hat{\epsilon}}(\tilde{s}) = 0 = \phi_{P_1,\hat{\epsilon}}(\tilde{s})$ and $\psi_{P_1,\hat{\epsilon}}(\tilde{s}) = 1$ that is,
$\emptyset_{FrNS,P_1} = \{(\hat{\epsilon}, \langle\tilde{s}, (0, 0, 1)\rangle : \tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P_1\}$.

5.6. Relative whole Fermatean Neutrosophic Soft set

$U_{FrNS,P_1} = \{(\hat{\epsilon}, \langle\tilde{s}, (1, 1, 0)\rangle : \tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P_1\}$.

5.7. Absolute Null Fermatean Neutrosophic Soft set

A $FrNSS$ $\mathfrak{X}_{FrNS,P}$ is considered as absolute null $FrNSS$ ($\emptyset_{FrNS,P}$) if for all $\hat{\epsilon} \in P, \theta_{P,\hat{\epsilon}}(\tilde{s}) = 0 = \phi_{P,\hat{\epsilon}}(\tilde{s})$ and $\psi_{P,\hat{\epsilon}}(\tilde{s}) = 1$ that is,
$\emptyset_{FrNS,P} = \{(\hat{\epsilon}, \langle\tilde{s}, (0, 0, 1)\rangle : \tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P\}$.

5.8. Absolute whole Fermatean Neutrosophic Soft set

A $FrNSS$ $\mathfrak{X}_{FrNS,P}$ is considered absolute whole $FrNSS$ ($U_{FrNS,\hat{\epsilon}}$) if for all $\hat{\epsilon} \in P, \tilde{s} \in \mathbb{D}, \theta_{P,\hat{\epsilon}}(\tilde{s}) = 1 = \phi_{P,\hat{\epsilon}}(\tilde{s})$ and $\psi_{P,\hat{\epsilon}}(\tilde{s}) = 0$ that is,
$U_{FrNS,P} = \{(\hat{\epsilon}, \langle\tilde{s}, (1, 1, 0)\rangle : \tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P\}$.

5.9. Proposition

Let $\mathfrak{X}_{FrNS,P_1}, \mathfrak{X}_{FrNS,P_2}, \mathfrak{X}_{FrNS,P_3}$ be $FrNSS$s, then
(i) $\emptyset_{FrNS,P_1} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1}$,
(ii) $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\subseteq} U_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\subseteq} U_{FrNS,P}$,
(iii) $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\tilde{\subseteq}} \emptyset_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\tilde{\subseteq}} \emptyset_{FrNS,P}$,
(iv) $U_{FrNS,P_1} \underset{FrNS}{\tilde{\subseteq}} \Xi_{FrNS,P_1}$,
(v) $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_2}$ and $\mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_3}$ implies $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_3}$,

(vi) $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\tilde{\subseteq}} \mathfrak{X}_{FrNS,P_2}$ and $\mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\tilde{\subseteq}} \mathfrak{X}_{FrNS,P_3}$ implies $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\tilde{\subseteq}} \mathfrak{X}_{FrNS,P_3}$,
(vii)$\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{=} \mathfrak{X}_{FrNS,P_2}$ and $\mathfrak{X}_{FrNS,P_2} \underset{FrNS}{=} \mathfrak{X}_{FrNS,P_3}$ implies $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{=} \mathfrak{X}_{FrNS,P_3}$.

5.9.1. Remark

Observe that $\emptyset_{FrNS,P} \underset{FrNS}{\not\subseteq} \mathfrak{X}_{FrNS,P_1}$ as $P \not\subseteq P_1$ and hence first condition of being $FrNS$ subset does not hold.

5.10. Not set of set of parameters

$\neg P = \{\neg\hat{\epsilon} : \hat{\epsilon} \in P, \neg\hat{\epsilon}$=not $\hat{\epsilon}\}$ is representing the not set of set of parameters P.

5.11. Complement of Fermatean Neutrosophic Soft Set

The complement of a $FrNSS$ $\mathfrak{X}_{FrNS,P_1}$, denoted by $\mathfrak{X}^c_{FrNS,P_1}$ is a $FrNSS$ given as $(f^{*c}, \neg P_1)$ where $f^{*c} : \neg P_1 \to \mathbb{P}(\mathbb{D})_{FrN}$ such that $\theta_{\neg P_1,\neg\hat{\epsilon}} = \psi_{P_1,\hat{\epsilon}}, \phi_{\neg P_1,\neg\hat{\epsilon}} = 1 - \phi_{P_1,\hat{\epsilon}}$ and $\psi_{\neg P_1,\neg\hat{\epsilon}} = \theta_{P_1,\hat{\epsilon}}$.

5.11.1. Example

The complement of $FrNSS$ $\mathfrak{X}_{FrNS,P_1}$ in example 5.1.1. is,
$\mathfrak{X}_{FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.1, 0.6, 0.8)\rangle, \langle\tilde{s}_2, (0.3, 0.3, 0.9)\rangle, \langle\tilde{s}_3, (0.3, 0.8, 0.1)\rangle)$,
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.4, 0.8, 0.6)\rangle, \langle\tilde{s}_2, (0.3, 0.3, 0.8)\rangle, \langle\tilde{s}_3, (0.7, 0.5, 0.1)\rangle)\}$.

5.12. Proposition

Let $\mathfrak{X}_{FrNS,P_1}$ be a $FrNSS$, then
(i) $(\mathfrak{X}^c_{FrNS,P_1})^c = \mathfrak{X}_{FrNS,P_1}$,
(ii) $\emptyset^c_{FrNS,P_1} = U_{FrNS,P_1}$,
(iii) $\emptyset^c_{FrNS,P} = U_{FrNS,P}$,
(iv) $U^c_{FrNS,P_1} = \emptyset_{FrNS,P_1}$,
(v) $U^c_{FrNS,P} = \emptyset_{FrNS,P}$.

5.13. Extended Union of Fermatean Neutrosophic Soft Sets

The extended union $(\cup_E)$ of two $FrNSS$s is a $FrNSS$ $\mathfrak{X}_{FrNS,P_3}$ where $P_3 = P_1 \cup P_2$ with associateship, indeterminacy, non-associateship for $\hat{\epsilon} \in P_1 \cup P_2$ is defined as follows,

$$\theta_{P_3,\hat{\epsilon}}, \phi_{P_3,\hat{\epsilon}}, \psi_{P_3,\hat{\epsilon}} = \begin{cases} \theta_{P_1,\hat{\epsilon}}, \phi_{P_1,\hat{\epsilon}}, \psi_{P_1,\hat{\epsilon}} & \text{if } \hat{\epsilon} \in P_1 \setminus P_2 \\ \theta_{P_2,\hat{\epsilon}}, \phi_{P_2,\hat{\epsilon}}, \psi_{P_2,\hat{\epsilon}} & \text{if } \hat{\epsilon} \in P_2 \setminus P_1 \\ \max\{\theta_{P_1,\hat{\epsilon}}, \theta_{P_2,\hat{\epsilon}}\}, \max\{\phi_{P_1,\hat{\epsilon}}, \phi_{P_2,\hat{\epsilon}}\}, \min\{\psi_{P_1,\hat{\epsilon}}, \psi_{P_2,\hat{\epsilon}}\} & \text{if } \hat{\epsilon} \in P_1 \cap P_2 \end{cases}$$

5.14. Restricted Union of Fermatean Neutrosophic Soft Sets

The restricted union $(\cup_R)$ of two $FrNSS$s is a $FrNSS$ $\mathfrak{X}_{FrNS,P_3}$ where $P_3 = P_1 \cap P_2$ with associateship, indeterminacy, non-associateship for $\hat{\epsilon} \in P_1 \cap P_2$ is defined as follows,
$\theta_{P_3,\hat{\epsilon}} = \max\{\theta_{P_1,\hat{\epsilon}}, \theta_{P_2,\hat{\epsilon}}\}, \phi_{P_3,\hat{\epsilon}} = \max\{\phi_{P_1,\hat{\epsilon}}, \phi_{P_2,\hat{\epsilon}}\}, \psi_{P_3,\hat{\epsilon}} = \min\{\psi_{P_1,\hat{\epsilon}}, \psi_{P_2,\hat{\epsilon}}\}$.

5.14.1. Example

Consider the $FrNSS$s $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ in example 5.1.1. and 5.2.1., respectively. Then their union will be,
$\mathfrak{X}_{FrNS,P_1} \cup_E \mathfrak{X}_{FrNS,P_2}$=$\{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.9, 0.5, 0.1)\rangle, \langle\tilde{s}_2, (0.9, 0.8, 0.1)\rangle, \langle\tilde{s}_3, (0.4, 0.5, 0.2)\rangle), (\hat{\epsilon}_2, \langle\tilde{s}_1, (0.7, 0.3, 0.2)\rangle, \langle\tilde{s}_2, (0.8, 0.7,$
$(\hat{\epsilon}_4, \langle\tilde{s}_1, (0.5, 0.2, 0.6)\rangle, \langle\tilde{s}_2, (0.7, 0.3, 0.5)\rangle, \langle\tilde{s}_3, (0.4, 0.2, 0.6)\rangle)\}$.
$\mathfrak{X}_{FrNS,P_1} \cup_R \mathfrak{X}_{FrNS,P_2}$=$\{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.9, 0.5, 0.1)\rangle, \langle\tilde{s}_2, (0.9, 0.8, 0.1)\rangle, \langle\tilde{s}_3, (0.4, 0.5, 0.2)\rangle), (\hat{\epsilon}_2, \langle\tilde{s}_1, (0.7, 0.3, 0.2)\rangle, \langle\tilde{s}_2, (0.8, 0.7,$

5.14.2. Remark

It is a clear observation that for any two $FrNSS$s $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$,
$\mathfrak{X}_{FrNS,P_1} \cup_R \mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1} \cup_E \mathfrak{X}_{FrNS,P_2}$.

5.15. Extended Intersection of Fermatean Neutrosophic Soft Sets

The extended intersection $(\cap_E)$ of two $FrNSS$s is a $FrNSS$ $\mathfrak{X}_{FrNS,P_3}$ where $P_3 = P_1 \cup P_2$ with associateship, indeterminacy, non-associateship for $\hat{\epsilon} \in P_1 \cup P_2$ is defined as follows,

$$\theta_{P_3,\hat{\epsilon}}, \phi_{P_3,\hat{\epsilon}}, \psi_{P_3,\hat{\epsilon}} = \begin{cases} \theta_{P_1,\hat{\epsilon}}, \phi_{P_1,\hat{\epsilon}}, \psi_{P_1,\hat{\epsilon}} & \text{if } \hat{\epsilon} \in P_1 \setminus P_2 \\ \theta_{P_2,\hat{\epsilon}}, \phi_{P_2,\hat{\epsilon}}, \psi_{P_2,\hat{\epsilon}} & \text{if } \hat{\epsilon} \in P_2 \setminus P_1 \\ \min\{\theta_{P_1,\hat{\epsilon}}, \theta_{P_2,\hat{\epsilon}}\}, \min\{\phi_{P_1,\hat{\epsilon}}, \phi_{P_2,\hat{\epsilon}}\}, \max\{\psi_{P_1,\hat{\epsilon}}, \psi_{P_2,\hat{\epsilon}}\} & \text{if } \hat{\epsilon} \in P_1 \cap P_2 \end{cases}$$

5.16. Restricted intersection of Fermatean Neutrosophic Soft Sets

The restricted intersection $(\cap_R)$ of two $FrNSS$s is a $FrNSS$ $\mathfrak{X}_{FrNS,P_3}$ where $P_3 = P_1 \cap P_2$ with associateship, indeterminacy, non-associateship $\hat{\epsilon} \in P_1 \cap P_2$ is defined as follows
$\theta_{P_3,\hat{\epsilon}} = \min\{\theta_{P_1,\hat{\epsilon}}, \theta_{P_2,\hat{\epsilon}}\}, \phi_{P_3,\hat{\epsilon}} = \min\{\phi_{P_1,\hat{\epsilon}}, \phi_{P_2,\hat{\epsilon}}\}, \psi_{P_3,\hat{\epsilon}} = \max\{\psi_{P_1,\hat{\epsilon}}, \psi_{P_2,\hat{\epsilon}}\}$.

5.16.1. Example

Consider the $FrNSS$s $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ in example 5.1.1. and 5.2.1., respectively. Then their intersection will be,
$\mathfrak{X}_{FrNS,P_1} \cap_E \mathfrak{X}_{FrNS,P_2}$=$\{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.8, 0.4, 0.1)\rangle, \langle\tilde{s}_2, (0.9, 0.7, 0.3)\rangle, \langle\tilde{s}_3, (0.1, 0.2, 0.3)\rangle), (\hat{\epsilon}_2, \langle\tilde{s}_1, (0.6, 0.2, 0.4)\rangle, \langle\tilde{s}_2, (0.8, 0.$
$(\hat{\epsilon}_4, \langle\tilde{s}_1, (0.5, 0.2, 0.6)\rangle, \langle\tilde{s}_2, (0.7, 0.3, 0.5)\rangle, \langle\tilde{s}_3, (0.4, 0.2, 0.6)\rangle)\}$.
$\mathfrak{X}_{FrNS,P_1} \cap_R \mathfrak{X}_{FrNS,P_2}$=$\{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.8, 0.4, 0.1)\rangle, \langle\tilde{s}_2, (0.9, 0.7, 0.3)\rangle, \langle\tilde{s}_3, (0.1, 0.2, 0.3)\rangle), (\hat{\epsilon}_2, \langle\tilde{s}_1, (0.6, 0.2, 0.4)\rangle, \langle\tilde{s}_2, (0.8, 0.$

5.16.2. Remark

It is a clear observation that for any two $FrNSS$s $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$,
$\mathfrak{X}_{FrNS,P_1} \cap_R \mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1} \cap_E \mathfrak{X}_{FrNS,P_2}$.

5.17. Proposition

Let $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ be two $FrNSS$s, then
(i) $\mathfrak{X}_{FrNS,P_1} \cap_R \mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1}, \mathfrak{X}_{FrNS,P_2}$,
(ii) $\mathfrak{X}_{FrNS,P_1}, \mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\tilde{\subseteq}} \mathfrak{X}_{FrNS,P_1} \cap_E \mathfrak{X}_{FrNS,P_2}$,
(iii) $\mathfrak{X}_{FrNS,P_1} \cup_R \mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\tilde{\subseteq}} \mathfrak{X}_{FrNS,P_1}, \mathfrak{X}_{FrNS,P_2}$,
(iv) $\mathfrak{X}_{FrNS,P_1}, \mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1} \cup_E \mathfrak{X}_{FrNS,P_2}$,
(v) $\mathfrak{X}_{FrNS,P_1} \cap_E \mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1} \cup_E \mathfrak{X}_{FrNS,P_2}$,
(vi) $\mathfrak{X}_{FrNS,P_1} \cap_R \mathfrak{X}_{FrNS,P_2} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1} \cup_R \mathfrak{X}_{FrNS,P_2}$,
(vii) $\mathfrak{X}_{FrNS,P_1} \cap_R \emptyset_{FrNS,P_1} = \emptyset_{FrNS,P_1}$,
(viii) $\mathfrak{X}_{FrNS,P_1} \cap_E \emptyset_{FrNS,P} = \emptyset_{FrNS,P}$,
(ix) $\mathfrak{X}_{FrNS,P_1} \cup_R U_{FrNS,P_1} = U_{FrNS,P_1}$,
(x) $\mathfrak{X}_{FrNS,P_1} \cup_E U_{FrNS,P} = U_{FrNS,P}$,
(xi) $\mathfrak{X}_{FrNS,P} \cap_R U_{FrNS,P} = \mathfrak{X}_{FrNS,P}$,
(xii) $\mathfrak{X}_{FrNS,P} \cap_E U_{FrNS,P} = \mathfrak{X}_{FrNS,P}$,
(xiii) $\mathfrak{X}_{FrNS,P} \cup_E \emptyset_{FrNS,P} = \mathfrak{X}_{FrNS,P}$,
(xiv) $\mathfrak{X}_{FrNS,P} \cup_R \emptyset_{FrNS,P} = \mathfrak{X}_{FrNS,P}$,
(xv) $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_2} \implies \mathfrak{X}_{FrNS,P_1} \cap_R \mathfrak{X}_{FrNS,P_2} = \mathfrak{X}_{FrNS,P_1}$,
(xvi) $\mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_2} \implies \mathfrak{X}_{FrNS,P_1} \cup_E \mathfrak{X}_{FrNS,P_2} = \mathfrak{X}_{FrNS,P_2}$,
(xvii) $(\mathfrak{X}_{FrNS,P_1} * \mathfrak{X}_{FrNS,P_2})^c = \mathfrak{X}^c_{FrNS,P_1} \hat{*} \mathfrak{X}^c_{FrNS,P_2}$, where $*, \hat{*} = \cup_R, \cup_E, \cap_R, \cap_E$, (De Morgan's law)
(xviii) $\mathfrak{X}_{FrNS,P_1} * (\mathfrak{X}_{FrNS,P_2} \hat{*} \mathfrak{X}_{FrNS,P_3}) = (\mathfrak{X}_{FrNS,P_1} * \mathfrak{X}_{FrNS,P_2}) \hat{*} (\mathfrak{X}_{FrNS,P_1} * \mathfrak{X}_{FrNS,P_2})$, where $*, \hat{*} = \cup_R, \cup_E, \cap_R, \cap_E$. (Distributive law)

6. ALGEBRAIC STRUCTURES

Algebraic structure is a set along with some operation/s or function satisfying a set of axioms. For example semi group, group, ring, field, vector space, metric space and normed space etc.
A semigroup under a binary operation $*$ is an algebraic structure satisfying closure and associative property

Table 4. Nomenclature for algebraic structures

Notation	Algebraic sructure
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, *\right)$	Semigroup
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, *\right)$	Monoid
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, *, \tilde{*}\right)$	Semiring

Table 5. Semigroups and Subsemigroups

semigroup	subsemigroups
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cup_E\right)$	$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_{P_1}, \cup_E\right)$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cup_R\right)$	$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_{P_1}, \cup_R\right)$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cap_E\right)$	$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_{P_1}, \cap_E\right)$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cap_R\right)$	$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_{P_1}, \cap_R\right)$

under $*$, while monoid is a semigroup with identity element under $*$ and semiring under $*, \hat{*}$ is the algebraic structure having following properties,
(i) commutative monoid under $*$, (ii) monoid under $\hat{*}$, (iii) distributive laws hold, (iv) operating $\hat{*}$ to identity element under $*$ and any element of considered set turns back to identity element.

Let $\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P$ and $\underset{FrNS}{\mathbb{P}}(\mathbb{D})_{P_1}$ be the collections of $FrNSS$s over the domain $\mathbb{D}$ associated with set of parameters P and a subset P_1 of P, respectively. The algebraic structures associated with $\cap_E, \cap_R, \cup_E$ and $\cup_R$ are established in tables 5, 6, 7. Table 4 is representing the nomenclature for defined algebraic structures.

7. FERMATEAN NEUTROSOPHIC SOFT TOPOLOGICAL SPACE

In this section, fermatean neutrosophic soft topological space ($FrNSTS$) is established defining the fermatean neutrosophic soft topology ($FrNST$).

7.1. Definition

Let $FrNSS(\mathbb{D}, P_1)$ be a collection of all $FrNSS$s over the domain of discourse $\mathbb{D}$ and set of parameters P_1. A subset $\tau_{frnsp_1} = \{\mathfrak{X}_{iFrNS,P_1} : i \in I\}$ of $FrNSS(\mathbb{D}, P_1)$ is named as $FrNST$ if following axioms are satisfied,
(i) $\emptyset_{FrNS,P_1}, U_{FrNS,P_1} \in \tau_{frnsp_1}$,
(ii) for a finite subset I' of index set I, if $\mathfrak{X}_{iFrNS,P_1} \in \tau_{frnsp_1}$ for $i \in I'$ then $\underset{i\in I'}{\cap} \mathfrak{X}_{iFrNS,P_1} \in \tau_{frnsp_1}$
that is the intersection of finite number of $FrNSS$ in τ_{frnsp_1} also belongs to τ_{frnsp_1},
(iii) if $\mathfrak{X}_{iFrNS,P_1} \in \tau_{frnsp_1}$ for $i \in I$ then $\underset{i\in I}{\cup} \mathfrak{X}_{iFrNS,P_1} \in \tau_{frnsp_1}$
that is the union of any number of $FrNSS$ in τ_{frnsp_1} also belongs to τ_{frnsp_1}.
The triplet $(\mathbb{D}, P_1, \tau_{frnsp_1})$ is named as $FrNSTS$.

Table 6. Commutative monoids

Monoid	Identity Element
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cup_E\right)$	$\emptyset_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cup_R\right)$	$\emptyset_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cap_E\right)$	$U_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cap_R\right)$	$U_{FrNS,P}$

Table 7. Semirings

Semiring (commutative, idempotent)	for any $\mathfrak{X}_{FrNS,P} \in \underset{FrNS}{\mathbb{P}}(\mathbb{D})_P$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cup_E, \cap_E\right)$	$\mathfrak{X}_{FrNS,P} \cap_E \emptyset_{FrNS,P} = \emptyset_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cup_E, \cap_R\right)$	$\mathfrak{X}_{FrNS,P} \cap_R \emptyset_{FrNS,P} = \emptyset_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cap_E, \cup_R\right)$	$\mathfrak{X}_{FrNS,P} \cup_R U_{FrNS,P} = U_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cap_E, \cup_E\right)$	$\mathfrak{X}_{FrNS,P} \cup_E U_{FrNS,P} = U_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cap_R, \cup_R\right)$	$\mathfrak{X}_{FrNS,P} \cup_R U_{FrNS,P} = U_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cap_R, \cup_E\right)$	$\mathfrak{X}_{FrNS,P} \cup_E U_{FrNS,P} = U_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cup_R, \cap_R\right)$	$\mathfrak{X}_{FrNS,P} \cap_R \emptyset_{FrNS,P} = \emptyset_{FrNS,P}$
$\left(\underset{FrNS}{\mathbb{P}}(\mathbb{D})_P, \cup_R, \cap_E\right)$	$\mathfrak{X}_{FrNS,P} \cap_E \emptyset_{FrNS,P} = \emptyset_{FrNS,P}$

7.1.1. Example

Let $\mathbb{D} = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3\}, P = \{\hat{\epsilon}_1, \hat{\epsilon}_2, ..., \hat{\epsilon}_5\}$ and $P_1 = \{\hat{\epsilon}_1, \hat{\epsilon}_2\}$. Let $FrNSS(\mathbb{D}, P_1)$ be a collection of all $FrNSS$ over the domain of discourse $\mathbb{D}$ and set of parameters P_1. Then $(\mathbb{D}, P_1, \tau_{frnsp_1})$ is a $FrNSTS$ with $FrNST, \tau_{frnsp_1} = \{\emptyset_{FrNS,P_1}, U_{FrNS,P_1}, \mathfrak{X}_{1FrNS,P_1}, \mathfrak{X}_{2FrNS,P_1}, \mathfrak{X}_{3FrNS,P_1}, \mathfrak{X}_{4FrNS,P_1}\}$. Here
$\mathfrak{X}_{1FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.8, 0.4, 0.1)\rangle, \langle\tilde{s}_2, (0.9, 0.7, 0.3)\rangle, \langle\tilde{s}_3, (0.1, 0.2, 0.3)\rangle)$,
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.6, 0.2, 0.4)\rangle, \langle\tilde{s}_2, (0.8, 0.7, 0.3)\rangle, \langle\tilde{s}_3, (0.1, 0.5, 0.7)\rangle)\}$,
$\mathfrak{X}_{2FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.6, 0.1, 0.9)\rangle, \langle\tilde{s}_2, (0.7, 0.6, 0.8)\rangle, \langle\tilde{s}_3, (0.9, 0.2, 0.3)\rangle)$,
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.8, 0.5, 0.6)\rangle, \langle\tilde{s}_2, (0.7, 0.5, 0.6)\rangle, \langle\tilde{s}_3, (0.4, 0.6, 0.5)\rangle)\}$,
$\mathfrak{X}_{3FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.6, 0.1, 0.9)\rangle, \langle\tilde{s}_2, (0.7, 0.6, 0.8)\rangle, \langle\tilde{s}_3, (0.1, 0.2, 0.3)\rangle)$,
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.6, 0.2, 0.6)\rangle, \langle\tilde{s}_2, (0.7, 0.5, 0.6)\rangle, \langle\tilde{s}_3, (0.1, 0.5, 0.7)\rangle)\}$,
$\mathfrak{X}_{4FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.8, 0.4, 0.1)\rangle, \langle\tilde{s}_2, (0.9, 0.7, 0.3)\rangle, \langle\tilde{s}_3, (0.9, 0.2, 0.3)\rangle)$,
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.8, 0.5, 0.4)\rangle, \langle\tilde{s}_2, (0.8, 0.7, 0.3)\rangle, \langle\tilde{s}_3, (0.4, 0.6, 0.5)\rangle)\}$.
Clearly, all axioms of $FrNST$ are satisfied.
(i) $\emptyset_{FrNS,P_1}, U_{FrNS,P_1} \in \tau_{frnsp_1}$,
(ii) Intersection of all possible (non-trivial) finite collections of elements of τ_{frnsp_1} is in τ_{frnsp_1} as follows,
for all $i = 1, 2, 3, 4$, $\emptyset_{FrNS,P_1} \cap \mathfrak{X}_{iFrNS,P_1} = \emptyset_{FrNS,P_1}$ and $U_{FrNS,P_1} \cap \mathfrak{X}_{iFrNS,P_1} = \mathfrak{X}_{iFrNS,P_1}$. $\mathfrak{X}_{1FrNS,P_1} \cap \mathfrak{X}_{2FrNS,P_1} = \mathfrak{X}_{3FrNS,P_1}$, $\mathfrak{X}_{1FrNS,P_1} \cap \mathfrak{X}_{3FrNS,P_1} = \mathfrak{X}_{3FrNS,P_1}$, $\mathfrak{X}_{1FrNS,P_1} \cap \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{1FrNS,P_1}$, $\mathfrak{X}_{2FrNS,P_1} \cap \mathfrak{X}_{3FrNS,P_1} = \mathfrak{X}_{3FrNS,P_1}$, $\mathfrak{X}_{2FrNS,P_1} \cap \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{2FrNS,P_1}$ and $\mathfrak{X}_{3FrNS,P_1} \cap \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{3FrNS,P_1}$, $\mathfrak{X}_{1FrNS,P_1} \cap \mathfrak{X}_{2FrNS,P_1} \cap \mathfrak{X}_{3F}$
$\mathfrak{X}_{3FrNS,P_1}$,
$\mathfrak{X}_{1FrNS,P_1} \cap \mathfrak{X}_{2FrNS,P_1} \cap \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{3FrNS,P_1}$,
$\mathfrak{X}_{1FrNS,P_1} \cap \mathfrak{X}_{3FrNS,P_1} \cap \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{3FrNS,P_1}$, $\mathfrak{X}_{2FrNS,P_1} \cap \mathfrak{X}_{3FrNS,P_1} \cap \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{3FrNS,P_1}$,
$\mathfrak{X}_{1FrNS,P_1} \cap \mathfrak{X}_{2FrNS,P_1} \cap \mathfrak{X}_{3FrNS,P_1} \cap \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{3FrNS,P_1}$.
(iii) Union of all possible (non-trivial) collections of elements of τ_{frnsp_1} is in τ_{frnsp_1} as shown,
for all $i = 1, 2, 3, 4$, $\emptyset_{FrNS,P_1} \cup \mathfrak{X}_{iFrNS,P_1} = \mathfrak{X}_{iFrNS,P_1}$ and $U_{FrNS,P_1} \cup \mathfrak{X}_{iFrNS,P_1} = U_{FrNS,P_1}$.
$\mathfrak{X}_{1FrNS,P_1} \cup \mathfrak{X}_{2FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$, $\mathfrak{X}_{1FrNS,P_1} \cup \mathfrak{X}_{3FrNS,P_1} = \mathfrak{X}_{1FrNS,P_1}$, $\mathfrak{X}_{1FrNS,P_1} \cup \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$, $\mathfrak{X}_{2FrNS,P_1} \cup \mathfrak{X}_{3FrNS,P_1} = \mathfrak{X}_{2FrNS,P_1}$, $\mathfrak{X}_{2FrNS,P_1} \cup \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$ and $\mathfrak{X}_{3FrNS,P_1} \cup \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$, $\mathfrak{X}_{1FrNS,P_1} \cup \mathfrak{X}_{2FrNS,P_1} \cup \mathfrak{X}_{3FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$,
$\mathfrak{X}_{1FrNS,P_1} \cup \mathfrak{X}_{2FrNS,P_1} \cup \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$,
$\mathfrak{X}_{1FrNS,P_1} \cup \mathfrak{X}_{3FrNS,P_1} \cup \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$, $\mathfrak{X}_{2FrNS,P_1} \cup \mathfrak{X}_{3FrNS,P_1} \cup \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$,
$\mathfrak{X}_{1FrNS,P_1} \cup \mathfrak{X}_{2FrNS,P_1} \cup \mathfrak{X}_{3FrNS,P_1} \cup \mathfrak{X}_{4FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$.

7.2. Indiscrete and Discrete Fermatean Neutrosophic Soft Topology

For $FrNSS(\mathbb{D}, P_1), \tau_{frnsp_1} = \{\emptyset_{FrNS,P_1}, U_{FrNS,P_1}\}$ is named as indiscrete $FrNST$ and $\tau_{frnsp_1} = FrNSS(\mathbb{D}, P_1)$ is named as discrete $FrNST$.

7.3. Coarser and Finer Fermatean Neutrosophic Soft Topology

More than one $FrNST$s could be defined over $\mathbb{D}$ with respect to the set of parameters P_1 and let $\tau^1_{frnsp_1}$ and $\tau^2_{frnsp_1}$ be two such $FrNST$s such that $\tau^1_{frnsp_1} \subset \tau^2_{frnsp_1}$. Then, $\tau^1_{frnsp_1}$ is named as coarser (smaller or weaker) $FrNST$ than $\tau^2_{frnsp_1}$ and $\tau^2_{frnsp_1}$ is named as finer (larger or stronger) $FrNST$ than $\tau^1_{frnsp_1}$.

7.3.1. Example

Consider the $FrNST$ τ_{frnsp_1} given in example 7.1.1. and let $\tau^1_{frnsp_1} = \{\emptyset_{FrNS,P_1}, U_{FrNS,P_1}, \mathfrak{X}_{1FrNS,P_1}\}$ be another $FrNST$ over the same domain of discourse $\mathbb{D}$ and set of parameters P_1. Clearly, $\tau^1_{frnsp_1} \subset \tau_{frnsp_1}$ and hence coarser $FrNST$ than τ_{frnsp_1} while τ_{frnsp_1} is finer $FrNST$ than $\tau^1_{frnsp_1}$.

7.4. Remark

Indiscrete $FrNST$ is the coarsest $FrNST$ while discrete $FrNST$ is the finest $FrNST$.

7.5. τ_{frnsp_1}-Open and τ_{frnsp_1}-Closed Fermatean Neutrosophic Soft Set

A $FrNSS$ $\mathfrak{X}_{FrNS,P_1}$ is named as τ_{frnsp_1}-open $FrNSS$ if it belongs to τ_{frnsp_1} and it is named as τ_{frnsp_1}-closed $FrNSS$ if $\mathfrak{X}^c_{FrNS,P_1}$ belongs to τ_{frnsp_1}.

7.5.1. Example

Consider the $FrNSTS$ $(\mathbb{D}, P_1, \tau_{frnsp_1})$ defined in example 7.1.1.. Here, $\mathfrak{X}_{1FrNS,P_1}$, $\mathfrak{X}_{2FrNS,P_1}, \mathfrak{X}_{3FrNS,P_1}, \mathfrak{X}_{4FrNS,P_1}$ are τ_{frnsp_1}-open $FrNSS$s while $\mathfrak{X}_{5FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.1, 0.6, 0.8)\rangle, \langle\tilde{s}_2, (0.3, 0.3, 0.9)\rangle, \langle$
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.4, 0.8, 0.6)\rangle, \langle\tilde{s}_2, (0.3, 0.3, 0.8)\rangle, \langle\tilde{s}_3, (0.7, 0.5, 0.1)\rangle\rangle)\}$,
$\mathfrak{X}_{6FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.9, 0.9, 0.6)\rangle, \langle\tilde{s}_2, (0.8, 0.4, 0.7)\rangle, \langle\tilde{s}_3, (0.3, 0.8, 0.9)\rangle\rangle)$,
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.6, 0.5, 0.8)\rangle, \langle\tilde{s}_2, (0.6, 0.5, 0.7)\rangle, \langle\tilde{s}_3, (0.5, 0.4, 0.4)\rangle\rangle)\}$,
$\mathfrak{X}_{7FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.9, 0.9, 0.6)\rangle, \langle\tilde{s}_2, (0.8, 0.4, 0.7)\rangle, \langle\tilde{s}_3, (0.3, 0.8, 0.1)\rangle\rangle)$,
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.6, 0.8, 0.6)\rangle, \langle\tilde{s}_2, (0.6, 0.5, 0.7)\rangle, \langle\tilde{s}_3, (0.7, 0.5, 0.1)\rangle\rangle)\}$,
$\mathfrak{X}_{8FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.1, 0.6, 0.8)\rangle, \langle\tilde{s}_2, (0.3, 0.3, 0.9)\rangle, \langle\tilde{s}_3, (0.3, 0.8, 0.9)\rangle\rangle)$,
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.4, 0.5, 0.8)\rangle, \langle\tilde{s}_2, (0.3, 0.3, 0.8)\rangle, \langle\tilde{s}_3, (0.5, 0.4, 0.4)\rangle\rangle)\}$ are τ_{frnsp_1}-closed $FrNSS$s. As the complement of these sets, $\mathfrak{X}^c_{5FrNS,P_1} = \mathfrak{X}_{1FrNS,P_1}, \mathfrak{X}^c_{6FrNS,P_1} = \mathfrak{X}_{2FrNS,P_1}, \mathfrak{X}^c_{7FrNS,P_1} = \mathfrak{X}_{3FrNS,P_1}, \mathfrak{X}^c_{8FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$ are in τ_{frnsp_1}.

7.6. Remark

For all i in an index set I, let $\tau^i_{frnsp_1}$ be $FrNST$s over $\mathbb{D}$ with respect to the set of parameters P_1. Then, $\bigcap_i \tau^i_{frnsp_1}$ is also a $FrNST$ over $\mathbb{D}$ with respect to the set of parameters P_1.

7.7. Fermatean Neutrosophic Soft Interior and Closure of a Fermatean Neutrosophic Soft Set

Let $\mathfrak{X}_{FrNS,P_1}$ be a $FrNSS$ in a $FrNSTS$ $(\mathbb{D}, P_1, \tau_{frnsp_1})$. The fermatean neutrosophic soft interior and closure of $\mathfrak{X}_{FrNS,P_1}$ are defined as follows,

$$\mathfrak{X}^o_{FrNS,P_1} = \cup\{A \in (\mathbb{D}, P_1, \tau_{frnsp_1}) : A \in \tau_{frnsp_1}, A \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1}\},$$
$$\bar{\mathfrak{X}}_{FrNS,P_1} = \cap\{A \in (\mathbb{D}, P_1, \tau_{frnsp_1}) : A^c \in \tau_{frnsp_1}, \mathfrak{X}_{FrNS,P_1} \underset{FrNS}{\subseteq} A\}.$$

Clearly, $\mathfrak{X}^o_{FrNS,P_1}$ is the union of τ_{frnsp_1}-open $FrNS$ subsets of $\mathfrak{X}_{FrNS,P_1}$ and $\bar{\mathfrak{X}}_{FrNS,P_1}$ is the intersection of τ_{frnsp_1}-closed $FrNS$ supersets of $\mathfrak{X}_{FrNS,P_1}$.

7.7.1. Example

Consider the $FrNSTS$ $(\mathbb{D}, P_1, \tau_{frnsp_1})$ defined in example 7.1.1.. The $FrNSS$, $\mathfrak{X}_{9FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.1, 0.5, 0.9)\rangle, \langle\tilde{s}_2, (0.2, 0.2, 0.9)\rangle, \langle\tilde{s}_3, (0.2, 0.6, 0.4)\rangle\rangle)$,
$(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.3, 0.6, 0.8)\rangle, \langle\tilde{s}_2, (0.2, 0.2, 0.9)\rangle, \langle\tilde{s}_3, (0.5, 0.4, 0.7)\rangle\rangle)\}$ is $FrNS$ subset of τ_{frnsp_1}-closed $FrNS$s $\mathfrak{X}_{5FrNS,P_1}$ and $\mathfrak{X}_{7FrNS,P_1}$. By definition, $\bar{\mathfrak{X}}_{9FrNS,P_1} = \mathfrak{X}_{5FrNS,P_1}$. Also, the $FrNSS$, $\mathfrak{X}_{10FrNS,P_1} = \{(\hat{\epsilon}_1, \langle\tilde{s}_1, (0.8, 0.5, 0.1)\rangle, \langle\tilde{s}_2, (0.9, 0.8, 0.2)\rangle$,
$\langle\tilde{s}_3, (0.9, 0.4, 0.2)\rangle\rangle)$, $(\hat{\epsilon}_2, \langle\tilde{s}_1, (0.9, 0.6, 0.2)\rangle, \langle\tilde{s}_2, (0.9, 0.8, 0.1)\rangle, \langle\tilde{s}_3, (0.7, 0.7, 0.4)\rangle\rangle)\}$ is $FrNS$ superset of τ_{frnsp_1}-open $FrNS$s $\mathfrak{X}_{3FrNS,P_1}$ and $\mathfrak{X}_{4FrNS,P_1}$. By definition, $\mathfrak{X}^o_{10FrNS,P_1} = \mathfrak{X}_{4FrNS,P_1}$.

7.8. Theorem

Let $\mathfrak{X}_{FrNS,P_1}$ be a $FrNSS$ in a $FrNSTS$ $(\mathbb{D}, P_1, \tau_{frnsp_1})$. Then,
(i) $\mathfrak{X}^o_{FrNS,P_1}$ is τ_{frnsp_1}-open $FrNSS$.
(ii) $\emptyset^o_{FrNS,P_1} = \emptyset_{FrNS,P_1}$, $U^o_{FrNS,P_1} = U_{FrNS,P_1}$ and $\mathfrak{X}^o_{FrNS,P_1} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1}$
(iii) $(\mathfrak{X}^o_{FrNS,P_1})^o = \mathfrak{X}^o_{FrNS,P_1}$
Proof
The proof is directly followed by definition.

7.9. Theorem

Let $\mathfrak{X}_{FrNS,P_1}$ be a $FrNSS$ in a $FrNSTS$ $(\mathbb{D}, P_1, \tau_{frnsp_1})$. Then,
(i) $\bar{\mathfrak{X}}_{FrNS,P_1}$ is τ_{frnsp_1}-closed $FrNSS$.
(ii) $\bar{\emptyset}_{FrNS,P_1} = \emptyset_{FrNS,P_1}$, $\bar{U}_{FrNS,P_1} = U_{FrNS,P_1}$ and $\bar{\mathfrak{X}}_{FrNS,P_1} \underset{FrNS}{\subseteq} \mathfrak{X}_{FrNS,P_1}$.
(iii) $\bar{\bar{\mathfrak{X}}}_{FrNS,P_1} = \bar{\mathfrak{X}}_{FrNS,P_1}$
(iv) $\mathfrak{X}_{FrNS,P_1}$ is τ_{frnsp_1}-closed $FrNSS$ if and only if $\bar{\mathfrak{X}}_{FrNS,P_1} = \mathfrak{X}_{FrNS,P_1}$ **Proof**
The proof is directly followed by definition.

7.10. Lemma

Let $\mathfrak{X}_{FrNS,P_1}$ be a $FrNSS$ in a $FrNSTS$ $(\mathbb{D}, P_1, \tau_{frnsp_1})$. Then,
(i) $(\mathfrak{X}^o_{FrNS,P_1})^c = \bar{\mathfrak{X}}^c_{FrNS,P_1}$,
(ii) $(\bar{\mathfrak{X}}_{FrNS,P_1})^c = (\mathfrak{X}^c_{FrNS,P_1})^o$,

Proof
Consider a $FrNSS$, $\mathfrak{X}_{FrNS,P_1}$ and let $\{A_i, i \in I\}$ be the collection of τ_{frnsp_1}-open $FrNS$ subsets of $\mathfrak{X}_{FrNS,P_1}$ defined as,
$A_i = \{(\hat{\epsilon}, \langle \tilde{s}, (\theta_{A_iP_1,\hat{\epsilon}}(\tilde{s}), \phi_{A_iP_1,\hat{\epsilon}}(\tilde{s}), \psi_{A_iP_1,\hat{\epsilon}}(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P_1\}$. Then, $\mathfrak{X}^o_{FrNS,P_1} = \{(\hat{\epsilon}, \langle \tilde{s}, \Big(\underset{i}{max}\, \theta_{A_iP_1,\hat{\epsilon}}(\tilde{s}), \underset{i}{max}\, \phi_{A_iP}$
$\tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P_1\}$ and $(\mathfrak{X}^o_{FrNS,P_1})^c = \{(\hat{\epsilon}, \langle \tilde{s}, \Big(\underset{i}{min}\, \psi_{A_iP_1,\hat{\epsilon}}(\tilde{s}), 1 - \underset{i}{max}\, \phi_{A_iP_1,\hat{\epsilon}}(\tilde{s}), \underset{i}{max}\, \theta_{A_iP_1,\hat{\epsilon}}(\tilde{s}) \Big)\rangle : \tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P_1\}$.
Clearly $\{A^c_i, i \in I\}$ is the collection of τ_{frnsp_1}-closed $FrNS$ supersets of $\mathfrak{X}^c_{FrNS,P_1}$ and $A^c_i = \{(\hat{\epsilon}, \langle \tilde{s}, (\psi_{A_iP_1,\hat{\epsilon}}(\tilde{s}), 1 - \phi_{A_iP_1,\hat{\epsilon}}($
$\tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P_1\}$. Then, $\bar{\mathfrak{X}}^c_{FrNS,P_1} = \{(\hat{\epsilon}, \langle \tilde{s}, \Big(\underset{i}{min}\, \psi_{A_iP_1,\hat{\epsilon}}(\tilde{s}), \underset{i}{min}\, (1 - \phi_{A_iP_1,\hat{\epsilon}})(\tilde{s}), \underset{i}{max}\, \psi_{A_iP_1,\hat{\epsilon}}(\tilde{s}) \Big)\rangle : \tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P_1\} = \{(\hat{\epsilon}, \langle \tilde{s}, \Big(\underset{i}{min}\, \psi_{A_iP_1,\hat{\epsilon}}(\tilde{s}), 1 - \underset{i}{max}\, \phi_{A_iP_1,\hat{\epsilon}}(\tilde{s}), \underset{i}{max}\, \psi_{A_iP_1,\hat{\epsilon}}(\tilde{s}) \Big)\rangle : \tilde{s} \in \mathbb{D}) : \hat{\epsilon} \in P_1\}$
and hence $(\mathfrak{X}^o_{FrNS,P_1})^c = \bar{\mathfrak{X}}^c_{FrNS,P_1}$.
The proof of (ii) is followed on similar lines.

7.11. Neighborhood of a Fermatean Neutrosophic Soft Set

Let $\mathfrak{X}_{1FrNS,P_1}$ be a $FrNSS$ in $(\mathbb{D}, P_1, \tau_{frnsp_1})$ then a $FrNSS$, $\mathfrak{X}_{2FrNS,P_1}$ is said to be a neighborhood of $\mathfrak{X}_{1FrNS,P_1}$ if $\mathfrak{X}_{2FrNS,P_1}$ is a τ_{frnsp_1}-open $FrNSS$ in $(\mathbb{D}, P_1, \tau_{frnsp_1})$ such that $\mathfrak{X}_{1FrNS,P_1} \underset{FrNS}{\subset} \mathfrak{X}_{2FrNS,P_1}$.

8. RELATION ON FERMATEAN NEUTROSOPHIC SOFT SET

In this section, relations on $FrNSS$ is established as that is used to develop decision making algorithm. $FrNS$ Relation is a $FrNS$ subset of cartesian product.

8.1. Cartesian Product

Let $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ be two $FrNSS$s. The cartesian product $\mathfrak{X}_{FrNS,P_1} \times \mathfrak{X}_{FrNS,P_2}$ of $FrNSS$s $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ is a $FrNSS$ $\mathfrak{X}_{FrNS,P_1 \times P_2} = \{(\xi, \langle \tilde{s}, (\theta_{P_1 \times P_2,\xi}(\tilde{s}), \phi_{P_1 \times P_2,\xi}(\tilde{s}), \psi_{P_1 \times P_2,\xi}(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}) : \xi \in P_1 \times P_2\}$ where $\theta_{P_1,\xi}, \phi_{P_1,\xi}, \psi_{P_1,\xi} : \mathbb{D} \to [0,1]$ such that for all $\tilde{s} \in \mathbb{D}$ and $\xi \in P_1 \times P_2, 0 \leq \theta^3_{P_1 \times P_2,\xi}(\tilde{s}) + \psi^3_{P_1 \times P_2,\xi}(\tilde{s}) \leq 1$ and $0 \leq \theta^3_{P_1 \times P_2,\xi}(\tilde{s}) + \phi^3_{P_1 \times P_2,\xi}(\tilde{s}) + \psi^3_{P_1 \times P_2,\xi}(\tilde{s}) \leq 2$, where
$\theta_{P_1 \times P_2} = \min\{\theta_{P_1}, \theta_{P_2}\}$, $\phi_{P_1 \times P_2} = \min\{\phi_{P_1}, \phi_{P_2}\}$ and $\psi_{P_1 \times P_2} = \max\{\psi_{P_1}, \psi_{P_2}\}$

8.2. Example

Let $\mathfrak{X}_{FrNS,P_1} = \{(\hat{\epsilon}_1, \langle \tilde{s}_1, (0.8, 0.4, 0.12) \rangle, \langle \tilde{s}_2, (0.9, 0.7, 0.3) \rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.35) \rangle)$,
$(\hat{\epsilon}_2, \langle \tilde{s}_1, (0.6, 0.27, 0.4) \rangle, \langle \tilde{s}_2, (0.83, 0.7, 0.3) \rangle, \langle \tilde{s}_3, (0.18, 0.52, 0.7) \rangle)\}$ and
$\mathfrak{X}_{FrNS,P_2} = \{(\hat{\epsilon}_2, \langle \tilde{s}_1, (0.7, 0.65, 0.3) \rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7) \rangle, \langle \tilde{s}_3, (0.29, 0.6, 0.37) \rangle)$,
$(\hat{\epsilon}_3, \langle \tilde{s}_1, (0.5, 0.74, 0.5) \rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63) \rangle, \langle \tilde{s}_3, (0.9, 0.3, 0.42) \rangle)\}$, then
$\mathfrak{X}_{FrNS,P_1 \times P_2} = \{((\hat{\epsilon}_1, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.7, 0.4, 0.3) \rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7) \rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.37) \rangle)$,
$((\hat{\epsilon}_1, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.4, 0.5) \rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63) \rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.42) \rangle)$,
$((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.6, 0.27, 0.4) \rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7) \rangle, \langle \tilde{s}_3, (0.18, 0.52, 0.7) \rangle)$,
$((\hat{\epsilon}_2, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.27, 0.5) \rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63) \rangle, \langle \tilde{s}_3, (0.18, 0.3, 0.7) \rangle)\}$.

8.3. Fermatean Neutrosophic Soft Relation

Let $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$ be two $FrNSS$s. A $FrNS$ relation from $\mathfrak{X}_{FrNS,P_1}$ to $\mathfrak{X}_{FrNS,P_2}$ is a $FrNS$ subset $\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}$ of $\mathfrak{X}_{FrNS,P_1 \times P_2}$, where $\mathbb{K} \times \mathbb{L} \subseteq P_1 \times P_2$.

8.4. Example

Consider the FrNS sets and their cartesian product in example 8.2.. Following are two FrNS relations between $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$,
$\mathbb{R}_{FrNS,P_1 \times P_2,1} = \mathfrak{X}_{FrNS,\mathbb{K}\times\mathbb{L}} =$
$\{((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.4, 0.2, 0.5) \rangle, \langle \tilde{s}_2, (0.3, 0.1, 0.9) \rangle, \langle \tilde{s}_3, (0, 0.3, 0.7) \rangle)$,
$((\hat{\epsilon}_2, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.2, 0.6) \rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.7) \rangle, \langle \tilde{s}_3, (0.1, 0.3, 0.75) \rangle)\}$, with $\mathbb{K} = \{\hat{\epsilon}_2\} \subseteq P_1$ and $\mathbb{L} = \{\hat{\epsilon}_2, \hat{\epsilon}_3\} \subseteq P_2$, and
$\mathbb{R}_{FrNS,P_1 \times P_2,2} = \mathfrak{X}_{FrNS,\mathbb{K}\times\mathbb{L}} =$
$((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.6, 0.27, 0.4) \rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7) \rangle, \langle \tilde{s}_3, (0.18, 0.52, 0.7) \rangle)$,
$((\hat{\epsilon}_2, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.27, 0.5) \rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63) \rangle, \langle \tilde{s}_3, (0.18, 0.3, 0.7) \rangle)\}$

8.5. Remark

As a relation from a set A with cardinality m to a set B with cardinality n is defined as a subset of cartesian product $A \times B$ so the number of possible relations from set A to set B is 2^{mn} but in case of Fermatean Neutrosophic soft set the number of $FrNS$ relations between two sets is more than the number of classical relations.

8.6. Domain and Range of Fermatean Neutrosophic Soft Relation

Let $\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}$ be FrNS relation from $\mathfrak{X}_{FrNS,P_1} = (f^*, P_1)$ to $\mathfrak{X}_{FrNS,P_2} = (g^*, P_2)$ then its domain and range is defined as,
Dom$(\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}) = (f^*|_{\mathbb{K}}, \mathbb{K}), \mathbb{K} \subseteq P_1$: for all $\hat{\epsilon}_i \in \mathbb{K}$, there exists $\hat{\epsilon}_j \in \mathbb{L}$ such that $(\hat{\epsilon}_i, \hat{\epsilon}_j) \in \mathbb{K} \times \mathbb{L}\}$
Range$(\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}) = (g^*|_{\mathbb{L}}, \mathbb{L}), \mathbb{L} \subseteq P_2$: for all $\hat{\epsilon}_j \in \mathbb{L}$, there exists $\hat{\epsilon}_i \in \mathbb{K}$ such that $(\hat{\epsilon}_i, \hat{\epsilon}_j) \in \mathbb{K} \times \mathbb{L}\}$

8.7. Example

In example 8.4., the domain and range of $\mathbb{R}_{FrNS,P_1 \times P_2,1}$ and $\mathbb{R}_{FrNS-P_1,P_2,2}$ are given as follows,
Dom$(\mathbb{R}_{FrNS,P_1 \times P_2,1})$ = Dom$(\mathbb{R}_{FrNS,P_1 \times P_2,2}) =$
$\{(\hat{\epsilon}_2, \langle \tilde{s}_1, (0.6, 0.27, 0.4) \rangle, \langle \tilde{s}_2, (0.83, 0.7, 0.3) \rangle, \langle \tilde{s}_3, (0.18, 0.52, 0.7) \rangle)\}$
Range$(\mathbb{R}_{FrNS,P_1 \times P_2,1})$ = Range$(\mathbb{R}_{FrNS,P_1 \times P_2,2}) =$
$\{(\hat{\epsilon}_2, \langle \tilde{s}_1, (0.7, 0.65, 0.3) \rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7) \rangle, \langle \tilde{s}_3, (0.29, 0.6, 0.37) \rangle)$,
$(\hat{\epsilon}_3, \langle \tilde{s}_1, (0.5, 0.74, 0.5) \rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63) \rangle, \langle \tilde{s}_3, (0.9, 0.3, 0.42) \rangle)\}$.

8.8. Inverse of a Fermatean Neutrosophic Soft Relation

Inverse of a FrNS relation $\mathbb{R}_{FrNS,\mathbb{K}\times\mathbb{L}}$ is $\mathbb{R}^{-1}_{FrNS,\mathbb{K}\times\mathbb{L}} = \mathbb{R}_{FrNS,\mathbb{L}\times\mathbb{K}}$.

8.9. Example

The inverse of FrNS relation $\mathbb{R}_{FrNS,P_1 \times P_2,1}$ in example 8.4. is,
$\mathbb{R}^{-1}_{FrNS,P_1 \times P_2,1}$=$\{((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.4, 0.2, 0.5) \rangle, \langle \tilde{s}_2, (0.3, 0.1, 0.9) \rangle, \langle \tilde{s}_3, (0, 0.3, 0.7) \rangle)$,
$((\hat{\epsilon}_3, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.5, 0.2, 0.6) \rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.7) \rangle, \langle \tilde{s}_3, (0.1, 0.3, 0.75) \rangle)\}$.

8.10. Composition of Fermatean Neutrosophic Soft Relations

Let $\mathbb{R}_{FrNS,P_1\times P_2} = \{\big(\xi_{ij} = (\hat{\epsilon}_i, \hat{\epsilon}_j), \langle \tilde{s}, \big(\theta_{P_1\times P_2,\xi_{ij}}(\tilde{s}), \phi_{P_1\times P_2,\xi_{ij}}(\tilde{s}), \psi_{P_1\times P_2,\xi_{ij}}(\tilde{s})\big)\rangle : \tilde{s} \in \mathbb{D}\big) :$ $\xi_{ij} \in P_1 \times P_2\}$ be a $FrNS$ relation from P_1 to P_2 and $\mathbb{R}_{FrNS,P_2\times P_3} = \{\big(\xi_{jk} = (\hat{\epsilon}_j, \hat{\epsilon}_k), \langle \tilde{s}, \big(\theta_{P_1\times P_2,\xi_{jk}}(\tilde{s}), \phi_{P_1\times P_2,\xi_{jk}}(\tilde{s}), \psi_{P_1\times}$ $\xi_{jk} \in P_2 \times P_3\}$ be $FrNS$ relation from P_2 to P_3 then composition of $\mathbb{R}_{FrNS-P_1,P_2}$ and $\mathbb{R}_{FrNS-P_2,P_3}$ is defined as,

$\mathbb{R}_{FrNS,P_1\times P_2} \circ \mathbb{R}_{FrNS,P_2\times P_3}$ =
$\{(\xi_{ik} = (\hat{\epsilon}_i, \hat{\epsilon}_k), \langle \tilde{s}, (\theta_{P_1\times P_3,\xi_{ik}}(\tilde{s}), \phi_{P_1\times P_3,\xi_{ik}}(\tilde{s}), \psi_{P_1\times P_3,\xi_{ik}}(\tilde{s}))\rangle : \tilde{s} \in \mathbb{D}) : \xi_{ik} \in P_1 \times P_3$
for which, there exist$(\hat{\epsilon}_i, \hat{\epsilon}_j) \in P_1 \times P_2and(\hat{\epsilon}_j, \hat{\epsilon}_k) \in P_2 \times P_3\}$, where
$\theta_{P_1\times P_3,\xi_{ik}} = \min\{\theta_{P_1\times P_2,\xi_{ij}}, \theta_{P_2\times P_3,\xi_{jk}}\}, \phi_{P_1\times P_3,\xi_{ik}} = \min\{\phi_{P_1\times P_2,\xi_{ij}}, \phi_{P_2\times P_3,\xi_{jk}}\}, \psi_{P_1\times P_3,\xi_{ik}} = \max\{\psi_{P_1\times P_2,\xi_{ij}}, \psi_{P_2\times P_3}$
that is $\mathbb{R}_{FrNS,P_1\times P_2} \circ \mathbb{R}_{FrNS,P_2\times P_3}(\hat{\epsilon}_i, \hat{\epsilon}_k) = \mathbb{R}_{FrNS,P_1\times P_2}(\hat{\epsilon}_i, \hat{\epsilon}_j) \cap_R \mathbb{R}_{FrNS,P_2\times P_3}(\hat{\epsilon}_j, \hat{\epsilon}_k)$

8.11. Example

Let $P_1 = \{\hat{\epsilon}_2\}, P_2 = \{\hat{\epsilon}_2, \hat{\epsilon}_3\}, P_3 = \{\hat{\epsilon}_4, \hat{\epsilon}_5\}$ be the set of parameters and consider the $FrNS$ relations, $\mathbb{R}_{FrNS,P_1\times P_2} = \{((\hat{\epsilon}_2, \hat{\epsilon}_2), \langle \tilde{s}_1, (0.4, 0.2, 0.5)\rangle,$
$\langle \tilde{s}_2, (0.3, 0.1, 0.9)\rangle, \langle \tilde{s}_3, (0, 0.3, 0.7)\rangle),$
$((\hat{\epsilon}_2, \hat{\epsilon}_3), \langle \tilde{s}_1, (0.5, 0.2, 0.6)\rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.7)\rangle, \langle \tilde{s}_3, (0.1, 0.3, 0.75)\rangle)\}$ and
$\mathbb{R}_{FrNS,P_2\times P_3} = \{((\hat{\epsilon}_2, \hat{\epsilon}_4), \langle \tilde{s}_1, (0.6, 0.2, 0.3)\rangle, \langle \tilde{s}_2, (0.5, 0.1, 0.7)\rangle, \langle \tilde{s}_3, (0.15, 0.2, 0.7)\rangle)$,
$((\hat{\epsilon}_2, \hat{\epsilon}_5), \langle \tilde{s}_1, (0.3, 0.52, 0.6)\rangle, \langle \tilde{s}_2, (0.46, 0.30.63)\rangle, \langle \tilde{s}_3, (0.5, 0.2, 0.77)\rangle)\}$, from P_1 to P_2 and from P_2 to P_3, respectively. Their composition is given by,
$\mathbb{R}_{FrNS,P_1\times P_2} \circ \mathbb{R}_{FrNS,P_2\times P_3} = \mathbb{R}_{FrNS,P_1\times P_3} =$
$\{((\hat{\epsilon}_2, \hat{\epsilon}_4), \langle \tilde{s}_1, (0.4, 0.2, 0.5)\rangle, \langle \tilde{s}_2, (0.3, 0.1, 0.9)\rangle, \langle \tilde{s}_3, (0, 0.2, 0.7)\rangle)$,
$((\hat{\epsilon}_2, \hat{\epsilon}_5), \langle \tilde{s}_1, (0.3, 0.2, 0.6)\rangle, \langle \tilde{s}_2, (0.46, 0.3, 0.7)\rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.77)\rangle)\}$.

8.12. Proposition

Let $\mathbb{R}_{FrNS,P_1\times P_2}$ and $\mathbb{R}_{FrNS,P_2\times P_3}$ be two $FrNS$ relations. Then
(i) $(\mathbb{R}^{-1}_{FrNS,P_1\times P_2})^{-1} = \mathbb{R}_{FrNS,P_1\times P_2}$,
(ii) $(\mathbb{R}_{FrNS,P_1\times P_2} \circ \mathbb{R}_{FrNS,P_2\times P_3})^{-1} = \mathbb{R}^{-1}_{FrNS,P_2\times P_3} \circ \mathbb{R}^{-1}_{FrNS,P_1\times P_2}$,
(iii) $\mathbb{R}_{FrNS,P_1\times P_2} \subseteq \mathbb{R}_{FrNS,P_2\times P_3}$ implies $\mathbb{R}^{-1}_{FrNS,P_1\times P_2} \subseteq \mathbb{R}^{-1}_{FrNS,P_2\times P_3}$.

This section presents a decision-making algorithm using fermatean neutrosophic soft relations. A sample problem is presented as an explanatory example.

9. SAMPLE PROBLEM

In a university, two friends A and B want to choose a common major for the bachelor's degree from a list of majors they both are interested in,
$\mathbb{D} = \{\text{Data Analytics, Information Technology, BSCS}\}$, according to their choice of parameters. Person A wants a major that assure the highly paid job oppertunaties and provide an exposure to real world application problems that is P_1={future employbility, best paying, exposure to real world applications} and person B wants a major that completes on time without any economic burden and associates with office-work jobs that is P_2={timely completion, economically efficient, office work job}. In our example problem, we have considered hypothetical data using Fermatean Neutrosophic set that could be replaced by the results of a survey. In order to choose a common major, we will take cartesian product of these sets to get all possible pairs of choices of A and B. By applying decision making approach, we will choose a major that accomodates the choices of both friends. Following figure shows the frame diagram for the stated problem.

9.1. Algorithm

The decision-making algorithm for our problem is explained in figure 2

Step I: Input the Fermatean Neutrosophic Soft sets.
$\mathfrak{X}_{FrNS,P_1} = \{(\hat{\epsilon}_1, \langle \tilde{s}_1, (0.7, 0.4, 0.3)\rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7)\rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.37)\rangle)$,
$(\hat{\epsilon}_2, \langle \tilde{s}_1, (0.5, 0.4, 0.5)\rangle, \langle \tilde{s}_2, (0.46, 0.5, 0.63)\rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.42)\rangle)$,
$(\hat{\epsilon}_3, \langle \tilde{s}_1, (0.6, 0.27, 0.4)\rangle, \langle \tilde{s}_2, (0.57, 0.2, 0.7)\rangle, \langle \tilde{s}_3, (0.18, 0.52, 0.7)\rangle)\}$. $\mathfrak{X}_{FrNS,P_2} = \{(\hat{\epsilon}_4, \langle \tilde{s}_1, (0.5, 0.27, 0.5)\rangle, \langle \tilde{s}_2, (0.46, 0.5,$
$(\hat{\epsilon}_5, \langle \tilde{s}_1, (0.8, 0.4, 0.12)\rangle, \langle \tilde{s}_2, (0.9, 0.7, 0.3)\rangle, \langle \tilde{s}_3, (0.1, 0.2, 0.35)\rangle)$,
$(\hat{\epsilon}_6, \langle \tilde{s}_1, (0.6, 0.27, 0.4)\rangle, \langle \tilde{s}_2, (0.83, 0.7, 0.3)\rangle, \langle \tilde{s}_3, (0.18, 0.52, 0.7)\rangle)\}$.

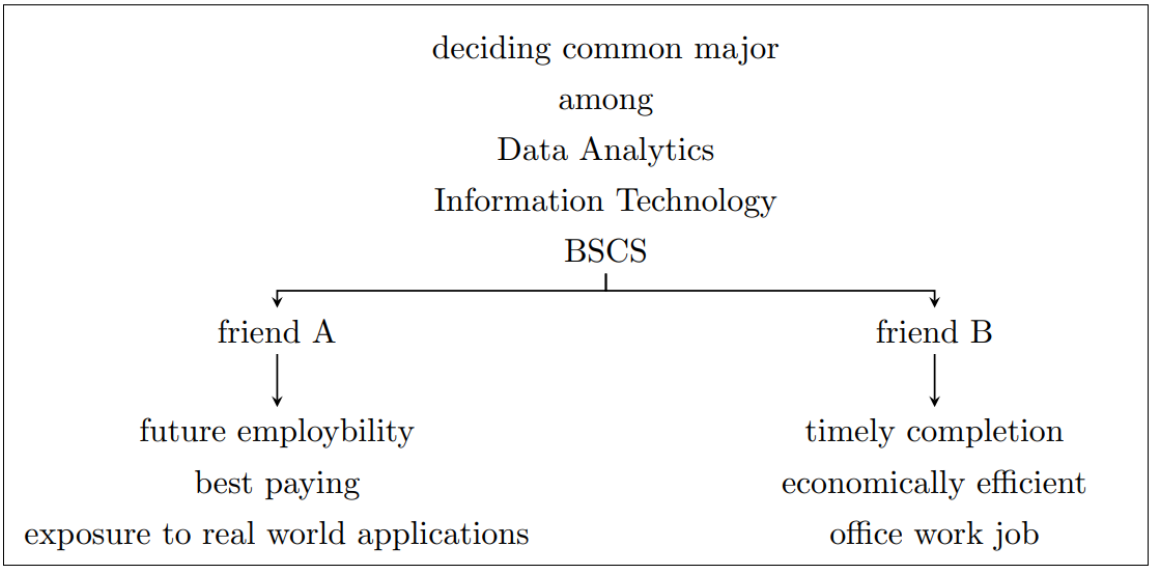

Figure 1. schematic representation of the problem

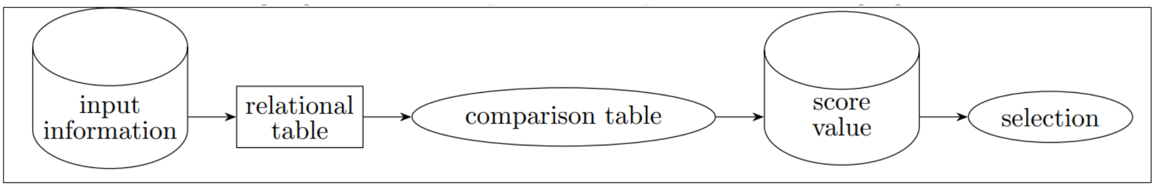

Figure 2. The decision making algorithm

Table 8. Information System

Parameter	$\tilde{s}_1$=Data Analytics	$\tilde{s}_2$=Information Technology	$\tilde{s}_3$=BSCS
$\hat{\epsilon}_1$= future employbility	$(0.7, 0.4, 0.3)$	$(0.57, 0.2, 0.7)$	$(0.1, 0.2, 0.37)$
$\hat{\epsilon}_2$= best paying	$(0.5, 0.4, 0.5)$	$(0.46, 0.5, 0.63)$	$(0.1, 0.2, 0.42)$
$\hat{\epsilon}_3$= exposure to real world applications	$(0.6, 0.27, 0.4)$ $(0.6, 0.27, 0.4)$	$(0.57, 0.2, 0.7)$ $(0.57, 0.2, 0.7)$	$(0.18, 0.52, 0.7)$ $(0.18, 0.52, 0.7)$
$\hat{\epsilon}_4$= timely completion	$(0.5, 0.27, 0.5)$	$(0.46, 0.5, 0.63)$	$(0.18, 0.3, 0.7)$
$\hat{\epsilon}_5$= economically efficient	$(0.8, 0.4, 0.12)$	$(0.9, 0.7, 0.3)$	$(0.1, 0.2, 0.35)$
$\hat{\epsilon}_6$= office work job	$(0.6, 0.27, 0.4)$	$(0.83, 0.7, 0.3)$	$(0.18, 0.52, 0.7)$

Table 9. Relational table between $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$

$(\hat{\epsilon}_i, \hat{\epsilon}_j)$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$
$(\hat{\epsilon}_1, \hat{\epsilon}_4)$	$(0.5, 0.27, 0.5)$	$(0.46, 0.2, 0.7)$	$(0.1, 0.2, 0.7)$
$(\hat{\epsilon}_1, \hat{\epsilon}_5)$	$(0.7, 0.4, 0.3)$	$(0.57, 0.2, 0.7)$	$(0.1, 0.2, 0.37)$
$(\hat{\epsilon}_1, \hat{\epsilon}_6)$	$(0.6, 0.27, 0.4)$	$(0.57, 0.2, 0.7)$	$(0.1, 0.2, 0.7)$
$(\hat{\epsilon}_2, \hat{\epsilon}_4)$	$(0.5, 0.27, 0.5)$	$(0.46, 0.5, 0.63)$	$(0.1, 0.2, 0.7)$
$(\hat{\epsilon}_2, \hat{\epsilon}_5)$	$(0.5, 0.4, 0.5)$	$(0.46, 0.5, 0.63)$	$(0.1, 0.2, 0.42)$
$(\hat{\epsilon}_2, \hat{\epsilon}_6)$	$(0.5, 0.27, 0.5)$	$(0.46, 0.5, 0.63)$	$(0.1, 0.2, 0.7)$
$(\hat{\epsilon}_3, \hat{\epsilon}_4)$	$(0.5, 0.27, 0.5)$	$(0.46, 0.2, 0.7)$	$(0.18, 0.2, 0.7)$
$(\hat{\epsilon}_3, \hat{\epsilon}_5)$	$(0.6, 0.27, 0.4)$	$(0.57, 0.2, 0.7)$	$(0.1, 0.2, 0.7)$
$(\hat{\epsilon}_3, \hat{\epsilon}_6)$	$(0.6, 0.27, 0.4)$	$(0.57, 0.2, 0.7)$	$(0.18, 0.52, 0.7)$

The corresponding information system is represented in the table 8. In the table, the first entry $(0.7, 0.4, 0.3)$ shows that the association of the parameter ”future employment” with the major ”Data Analytics” has associateship ”0.7”, indeterminacy ”0.4” and non-associateship ”0.3”.
Step II: Construct the Fermatean Neutrosophic Soft relational table as a result of their cartesian product as shown in table 9.
Step III: Contruct the coomparison table with the reference of teble 9 evaluating the value $\theta + \phi - \psi$ for each $(\hat{\epsilon}_i, \hat{\epsilon}_j)$ as shown in table 10.
Step IV: Calculate the score value by adding the highest value in each row against $\tilde{s}_i$, as shown in table 11.
Step V: Select the object with highest score value.
Both friends will choose ”Data Analytics” as major.

9.2. Remark

Above mentioned technique provides an algorithm for decision-making application. The formulas used for constructing comparison table and score function could be replaced by some other version of these e.g. mentioned in [59, 60]. Also if two or more objects get same score value one may apply accuracy function to get a precise decision [60].

Table 10. Comparison table between $\mathfrak{X}_{FrNS,P_1}$ and $\mathfrak{X}_{FrNS,P_2}$

$(\hat{\epsilon}_i, \hat{\epsilon}_j)$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$
$(\hat{\epsilon}_1, \hat{\epsilon}_4)$	0.27	$-.04$	0.4
$(\hat{\epsilon}_1, \hat{\epsilon}_5)$	0.8	0.07	-0.07
$(\hat{\epsilon}_1, \hat{\epsilon}_6)$	0.47	0.07	-0.4
$(\hat{\epsilon}_2, \hat{\epsilon}_4)$	0.27	0.33	-0.4
$(\hat{\epsilon}_2, \hat{\epsilon}_5)$	0.4	0.33	-0.12
$(\hat{\epsilon}_2, \hat{\epsilon}_6)$	0.27	0.33	-0.4
$(\hat{\epsilon}_3, \hat{\epsilon}_4)$	0.27	$-.04$	-0.32
$(\hat{\epsilon}_3, \hat{\epsilon}_5)$	0.47	0.07	-0.4
$(\hat{\epsilon}_3, \hat{\epsilon}_6)$	0.47	0.07	-0.32

Table 11. Score Value of Each Object

$\tilde{s}_i$	highest value from comaparison table	Score Value
$\tilde{s}_1$	$0.27 + 0.8 + 0.47 + 0.4 + 0.27 + 0.47 + 0.47$	3.15
$\tilde{s}_2$	$0.33 + 0.33$	0.66
$\tilde{s}_3$	0	0

10. CONCLUSION

The main motivation of this paper is to define a hybrid neutrosophic structure to get a wider possible range of neutrosophic numbers dealing with real-world problems. In this paper, a hybrid structure, fermatean Neutrosophic Soft set is defined along with the basic entities of soft set theory as $FrNS$ subset, absolute null $FrNSS$, relative null $FrNSS$, absolute whole $FrNSS$, relative whole $FrNSS$ as well as $FrNS$ twisted subset. A few operations as complement, extended and restricted intersection and union are defined. In section 6, algebraic structures as semigroups, subsemigroups, monoids and semirings are defined with respect to the operations defined on $FrNSS$s. Section 7 explores the definition and properties of fermatean neutrosophic soft topological spaces. Section 8 explores the relation defined on $FrNSS$s named as $FrNS$ relation being a $FrNS$ subset of cartesian product of $FrNSS$s. Section 9 deals with its application to decision-making problems using a decision making algorithm. This paper provides fundamentals of $FrNSS$ that act as a base to deal with different application problems and to define binary operations and algebraic srtucture with respect to the binary operations. Another possible extension could be $FrNS$ hypersoft set.

Funding: This research received no external funding.
Conflict of interest: The authors declare no conflicts of interest.

REFERENCES

[1] J. W. Dauben, *Georg Cantor: His mathematics and philosophy of the infinite*. Princeton University Press, 2020.

[2] L. A. Zadeh, "Fuzzy sets," *Information and control*, vol. 8, no. 3, pp. 338–353, 1965.

[3] Z. Pawłak, "Rough sets," *Int. J. Comput. Inf. Sci*, vol. 11, no. 5, pp. 341–356, 1982.

[4] D. Molodtsov, "Soft set theory—first results," *Computers & mathematics with applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[5] U. Acar, F. Koyuncu, and B. Tanay, "Soft sets and soft rings," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3458–3463, 2010.

[6] P. Maji, A. R. Roy, and R. Biswas, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[7] A. R. Roy and P. Maji, "A fuzzy soft set theoretic approach to decision making problems," *Journal of computational and Applied Mathematics*, vol. 203, no. 2, pp. 412–418, 2007.

[8] M. Saeed, S. Mahmood, and H. Taufeeq, "Hybridization of soft expert set with ahp in decision making," *International Journal of Computer Applications*, vol. 179, no. 4, pp. 24–29, 2017.

[9] H. Bustince and P. Burillo, "Vague sets are intuitionistic fuzzy sets," *Fuzzy sets and systems*, vol. 79, no. 3, pp. 403–405, 1996.

[10] K. T. Atanassov and K. T. Atanassov, *Intuitionistic fuzzy sets*. Springer, 1999.

[11] V. Torra and Y. Narukawa, "On hesitant fuzzy sets and decision," in *2009 IEEE international conference on fuzzy systems*. IEEE, 2009, pp. 1378–1382.

[12] K. Babitha and S. J. John, "Hesitant fuzzy soft sets," *Journal of New Results in Science*, vol. 2, no. 3, 2013.

[13] B. C. Cuong and V. Kreinovich, "Picture fuzzy sets-a new concept for computational intelligence problems," in *2013 third world congress on information and communication technologies (WICT 2013)*. IEEE, 2013, pp. 1–6.

[14] Y. Yang, C. Liang, S. Ji, and T. Liu, "Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 29, no. 4, pp. 1711–1722, 2015.

[15] M. Saeed, M. Ahsan, M. K. Siddique, and M. R. Ahmad, *A study of the fundamentals of hypersoft set theory*. Infinite Study, 2020.

[16] M. Saeed, M. Saqlain, A. Mehmood, and S. Yaqoob, "Multi-polar neutrosophic soft sets with application in medical diagnosis anddecision-making," *Neutrosophic Sets and Systems*, vol. 33, pp. 183–207, 2020.

[17] M. Saqlain, S. Moin, M. N. Jafar, M. Saeed, and F. Smarandache, *Aggregate operators of neutrosophic hypersoft set*. Infinite Study, 2020.

[18] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers & mathematics with applications*, vol. 45, no. 4-5, pp. 555–562, 2003.

[19] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir, "On some new operations in soft set theory," *Computers & Mathematics with Applications*, vol. 57, no. 9, pp. 1547–1553, 2009.

[20] N. Çağman and S. Enginoğlu, "Soft matrix theory and its decision making," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3308–3314, 2010.

[21] K. Babitha and J. Sunil, "Soft set relations and functions," *Computers & Mathematics with Applications*, vol. 60, no. 7, pp. 1840–1849, 2010.

[22] K. Babitha and J. J. Sunil, "Transitive closures and orderings on soft sets," *Computers & Mathematics with Applications*, vol. 62, no. 5, pp. 2235–2239, 2011.

[23] H.-L. Yang and Z.-L. Guo, "Kernels and closures of soft set relations, and soft set relation mappings," *Computers & mathematics with applications*, vol. 61, no. 3, pp. 651–662, 2011.

[24] A. Sezgin and A. O. Atagün, "On operations of soft sets," *Computers & Mathematics with Applications*, vol. 61, no. 5, pp. 1457–1467, 2011.

[25] X. Ge and S. Yang, "Investigations on some operations of soft sets," *World Academy of Science Engineering and Technology*, vol. 75, pp. 1113–1116, 2011.

[26] A. Sezgin and A. O. Atagün, "On operations of soft sets," *Computers & Mathematics with Applications*, vol. 61, no. 5, pp. 1457–1467, 2011.

[27] P. Zhu and Q. Wen, "Operations on soft sets revisited," *Journal of Applied Mathematics*, vol. 2013, no. 1, p. 105752, 2013.

[28] H. Aktaş and N. Çağman, "Soft sets and soft groups," *Information sciences*, vol. 177, no. 13, pp. 2726–2735, 2007.

[29] M. Aslam and S. M. Qurashi, "Some contributions to soft groups," *Ann. of fuzzy Math. and Inf*, vol. 4, pp. 177–195, 2012.

[30] F. Smarandache, *Neutrosophy: Neutrosophic Probability, Set, and Logic: Analytic Synthesis & Synthetic Analysis*. Rehoboth, NM: American Research Press, 1998.

[31] A. Azim, A. Ali, A. Khan, S. Ali, F. Awwad, and E. Ismail, "q-spherical fuzzy rough einstein geometric aggregation operator for image understanding and interpretations," *IEEE Access*, 2024.

[32] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, *Single valued neutrosophic sets*. Infinite study, 2010.

[33] R. Sunderraman and H. Wang, "Interval neutrosophic sets and logic: theory and applications in computing," *arXiv: Logic in Computer Science*, 2005.

[34] A. Salama and S. Alblowi, "Neutrosophic set and neutrosophic topological spaces," *Neutrosphic Sets and Systems*, 2012.

[35] M. Shabir and M. Naz, "On soft topological spaces," *Computers & Mathematics with Applications*, vol. 61, no. 7, pp. 1786–1799, 2011.

[36] T. Bera and N. K. Mahapatra, "Introduction to neutrosophic soft topological space," *Opsearch*, vol. 54, no. 4, pp. 841–867, 2017.

[37] F. Smarandache, *Neutrosophic measure and neutrosophic integral*. Infinite Study, 2013.

[38] V. Kandasamy and F. Smarandache, *Neutrosophic lattices*. Infinite Study, 2014.

[39] A. Agboola and S. Akinleye, "Neutrosophic vector spaces," *Neutrosophic Sets and Systems*, vol. 4, pp. 9–18, 2014.

[40] A. Salama, F. Smarandache, and V. Kroumov, "Neutrosophic closed set and neutrosophic continuous functions," *Neutrosophic sets and Systems*, vol. 4, pp. 4–8, 2014.

[41] V. Pătraşcu, *The neutrosophic entropy and its five components*. Infinite Study, 2015.

[42] A. AD and Y. Oyebo, "Neutrosophic groups and subgroups," *Mathematical Combinatorics*, vol. 3, pp. 1–9, 2012.

[43] M. Ali, F. Smarandache, M. Shabir, and M. Naz, "Soft neutrosophic ring and soft neutrosophic field," *Neutrosophic Sets and Systems*, 2014.

[44] A. Salama, F. Smarandache, and M. Eisa, "Introduction to image processing via neutrosophic techniques," *Infinite study*, 2014.

[45] S. Ye, J. Fu, and J. Ye, "Medical diagnosis using distance-based similarity measures of single valued neutrosophic multisets," *Neutrosophic Sets and Systems*, vol. 7, no. 1, pp. 47–52, 2015.

[46] A. U. Rahman, M. Saeed, M. A. Mohammed, S. Krishnamoorthy, S. Kadry, and F. Eid, "An integrated algorithmic madm approach for heart diseases' diagnosis based on neutrosophic hypersoft set with possibility degree-based setting," *Life*, vol. 12, no. 5, p. 729, 2022.

[47] A. N. H. Zaied, "Applications of fuzzy and neutrosophic logic in solving multi-criteria decision making problems," *Neutrosophic Sets and Systems*, vol. 13, pp. 38–46, 2016.

[48] M. N. Jafar, M. Saeed, K. M. Khan, F. S. Alamri, and H. A. E.-W. Khalifa, "Distance and similarity measures using max-min operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems," *Ieee Access*, vol. 10, pp. 11 220–11 235, 2022.

[49] M. R. Ahmad, M. Saeed, U. Afzal, and M.-S. Yang, "A novel mcdm method based on plithogenic hypersoft sets under fuzzy neutrosophic environment," *Symmetry*, vol. 12, no. 11, p. 1855, 2020.

[50] M. Saeed, A. U. Rahman, and M. Arshad, "A study on some operations and products of neutrosophic hypersoft graphs," *Journal of Applied Mathematics and Computing*, vol. 68, no. 4, pp. 2187–2214, 2022.

[51] T. Senapati and R. R. Yager, "Fermatean fuzzy sets," *Journal of ambient intelligence and humanized computing*, vol. 11, pp. 663–674, 2020.

[52] S. Broumi, S. Mohanaselvi, T. Witczak, M. Talea, A. Bakali, and F. Smarandache, "Complex fermatean neutrosophic graph and application to decision making," *Decision Making: Applications in Management and Engineering*, vol. 6, no. 1, pp. 474–501, 2023.

[53] N. Gonul Bilgin, D. Pamučar, and M. Riaz, "Fermatean neutrosophic topological spaces and an application of neutrosophic kano method," *Symmetry*, vol. 14, no. 11, p. 2442, 2022.

[54] V. Salsabeela and S. J. John, "Topsis techniques on fermatean fuzzy soft sets," in *American Institute of Physics Conference Series*, vol. 2336, no. 1, 2021, p. 040022.

[55] M. Goray and R. N. Annavarapu, "The energy of a photon, on the geometrical perspective," *Optik*, vol. 248, p. 168076, 2021.

[56] E. Goldfain, "Photon-neutrino symmetry and the opera anomaly: a neutrosophic viewpoint," *Florentin Smarandache*, vol. 60, no. 6.9, p. 82, 2011.

[57] F. Yuhua, "Neutrosophic examples in physics," *Neutrosophic Sets and Systems*, vol. 1, pp. 26–33, 2013.

[58] A. Hussain and M. Shabir, "Algebraic structures of neutrosophic soft sets," *Neutrosophic Sets and Systems*, vol. 7, no. 1, p. 10, 2015.

[59] P. K. Maji, "A neutrosophic soft set approach to a decision making problem," *Annals of Fuzzy mathematics and Informatics*, vol. 3, no. 2, pp. 313–319, 2012.

[60] P. Biswas, S. Pramanik, and B. C. Giri, "Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making," *Neutrosophic sets and systems*, vol. 12, pp. 20–40, 2016.