International Journal of Mathematics, Statistics, and Computer Science
Vol. 4
eISSN 2704-1069, DOI: https://doi.org/10.59543/r1yz0411

A Study on the Numerical Accuracy of Galerkin, Modified Galerkin, Shooting and Homotopy Perturbation Methods in Solving Boundary Value Problems

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Received 25.11.2025, Revised 18.02.2026, Accepted 20.02.2026

ABSTRACT: This study analyzes the behavior of four popular numerical methods of solving boundary-value problems (BVPs): the Standard Galerkin method, the Modified Galerkin method, the Shooting method, and the Homotopy Perturbation Method (HPM). The BVP is commonly used in scientific and engineering practice in fluid and thermal transport, micro- and bio-fluidic systems, fluid-structure interaction, aerodynamics, and electromagnetic modelling. The evaluation of each technique was based on the comparison of its numerical results and the errors with the analytical result. The precision and consistency were reflected by detailed tables and graphic representations that point out solution behaviour and error patterns. The best accuracy was derived with the Shooting method, followed by the Modified Galerkin method. The modified Galerkin method was more adaptable and stable as compared to the standard Galerkin scheme, which showed bigger variations in its error outcomes. HPM, however, was observed to have some irregularities, especially at the mid-point part of the solution domain. Therefore, the general comparison of this paper explains how each numerical technique reacts to boundary-value problems and how much they can be applied in real computational settings.

Keywords: Error Analysis, Boundary Value Problems (BVPs), Shooting Method, Ordinary Differential Equations (ODEs), Numerical Analysis.

1. INTRODUCTION

Computational mathematics acknowledges the important numerical treatment of ordinary differential and boundary value problems. These methods provide high accuracy and reliability of solutions; hence, the closure of the difference between the practical and theoretical uses [1, 2]. A very large number of phenomena in applied sciences, including engineering, physics, and others, are habitually formulated as a boundary value problem (BVP). Some of them are the study of stresses, strains, and modal vibrations in structural elements, the study of radiative cooling, and the study of heat transfer mechanisms, among others. In most cases, these systems are very complex and non-linear; thus, it is not possible to derive precise solutions. Accordingly, the use of powerful, effective numerical methods is inevitable, considering they have the potential to provide quality and fast estimations. Accordingly, researchers have been focusing seriously on elaborate schemes, which include the Galerkin method, the modified Galerkin method, the Shooting method, and the Homotopy Perturbation method (HPM).

A popular classical technique called the Shooting method converts a BVP into an IVP and iteratively modifies the initial parameters to satisfy the boundary constraints [3]. Convergence is contingent on the quality of the initial guess, and it can become unstable for highly non-linear problems [4]. Singh [5] studied the Shooting method for finding a solution to two-point BVPs of numerical problems. Kai Diethelm [6] uses the Shooting method for evaluating non-linear fractional BVPs. Md. M. Rahman [7] worked on the classical Shooting method to reduce two-point boundary-value problems to initial-value problems and solve them by using numerical integration, demonstrating with using illustrative examples that the technique allows to find accurate and efficient solutions to nonlinear boundary-value problems. Thomée [8] worked on Galerkin finite element methods for solving parabolic PDEs. G. Zavalani [9] used the classical Galerkin finite element approach to solve second-order two-point boundary-value problems, carefully deriving the corresponding weak formulation and assembling finite-element basis functions, to obtain high-fidelity solution to the problem. By making comparative analyses, with a series of representative examples, the proposed approach is shown to provide results that are both stable, and reliable with respect to a series of conventional numerical methods. N. S. Hussain et al. [10] formulate and discuss a weak Galerkin finite element method to solve second-order elliptic partial differential equations with mixed boundary conditions, proving their stability and convergence, and showing good numerical behavior. Yi Tian et al. [11] analysed the Galerkin method for evaluating some BVPs.

The modified Galerkin method is an improved version of the Galerkin method, which makes some calculations easier by shifting derivatives to another part of the method. Mitsotakis et al. [12] applied a modified Galerkin finite element method for solving the fully non-linear shallow water system using low-order elements. W. Yajni et al. [13] used a modified version of the Galerkin methodology to compute the solution of the Helmholtz equation at low frequencies considering the environmental parameters of Mars and thus improving both the convergence characteristics and the numerical accuracy of the method with respect to wave-propagation computations on Mars. The Homotopy Perturbation method (HPM) proposed by Ji-Huan He [14] is a semi-analytical procedure that combines the homotopy method and perturbation technique. Topman et al. [15] applied HPM to analyze the Influenza virus model and study how the influenza virus spreads in a population. S. Siddiqi et al. [16] used the Homotopy Perturbation method for multiple BVPs. E. Buhe et al. [17] used the HPM to explain how population affects forest depletion and gave a semi-analytical solution to a nonlinear mathematical model which provides an extremely good series representation.

Bizzarri et al. [18] expressed Galerkin projections with Shooting methods to improve the efficiency of the circuit simulations. Ahmad et al. [19] compared the Shooting method to the finite-difference method and showed that it was much more accurate on particular classes of BVPs. Almutairi et al. [20] compare the Galerkin finite element method(GFEM) and finite difference method(FDM) to highlight their benefits and drawbacks for resolving boundary value issues. D. Sharma et al. [21] compared a B-spline finite element method with the GFEM for a two-point BVP. Ibrahim et al. [22] analyze the efficiency of Shooting and FDMs for non-linear BVP. C. Chun et al. [23] express the effectiveness of the Shooting method, extended Adomian decomposition method, and Homotopy Perturbation approach for non-linear BVP in a finite domain. S. Momani et al. [24] compare the Differential transform method, Adomian decomposition, and HPM for fourth-order BVP.

Despite these contributions, a significant research gap remains. Previous studies mainly compared one or two techniques among Galerkin, Modified Galerkin, Shooting, and Homotopy Perturbation methods. No systematic study has compared all four approaches with computational efficiency as a primary factor. The effectiveness of divergent numerical methods depends on the specific features of the BVP. However, a rigorous comparison of these four approaches is required to elucidate differences in computational accuracy, error types, and distributions as well as efficiency across all conditions that can possibly occur.

The aim of this paper is, therefore, to fill in such a gap by performing a critical comparison among the Galerkin method, the modified Galerkin method, Shooting method, and the Homotopy Perturbation method. This work encourages practitioners to choose the best method, which depends on such criteria as accuracy, computation time, and the amount of required computations.

2. MATERIALS AND METHOD

In this section, a boundary value problem (BVP) is considered in the form:

$$\mathcal{L}[u(x)] = f(x), \; u(a) = \alpha, \; u(b) = \beta$$

where $\mathcal{L}$ is a linear or non-linear differential operator, and different numerical/analytical approximation methods are applied to solve and compare their effectiveness. The following methods are presented:

2.1 Galerkin Method

The Galerkin Method is a weighted residual approach [8]. The approximate solution is expressed as:

$$u(x) \approx \Sigma\, c_i\, \varphi_i(x), \; i = 1, \dots, n$$

where $\varphi_i(x)$ trial is functions satisfying boundary conditions. Substitution into the governing differential equation gives a residual $R(x)$:

$$R(x) = \mathcal{L}[u(x)] - f(x)$$

The Galerkin condition requires:

$$\int_a^b R(x)\,\varphi_j(x)\,dx = 0, \; j = 1, \dots, n$$

2.2 Modified Galerkin Method

The modified Galerkin method extends the classical Galerkin approach by introducing correction terms or modified weight functions to enhance convergence and accuracy [12]. The approximate solution takes the form:

$$u(x) \approx \Sigma\, c_i\, \psi_i(x), \; i = 1, \dots, n$$

where $\psi_i(x)$ are modified trial functions chosen to better capture boundary behavior. The weighted residual condition becomes:

$$\int_a^b R(x)\,w_j(x)\,dx = 0, \; j = 1, \dots, n$$

2.3 Shooting Method

The Shooting Method transforms the BVP into an IVP [4]. Assuming an unknown initial slope $u'(a) = s$, we solve:

$$u''(x) = g(x, u, u'), \; u(a) = \alpha, \; u'(a) = s$$

using a numerical IVP solver. The computed solution $\hat{u}(b; s)$ is compared with the boundary condition β. The shooting condition becomes:

$$\varphi(s) = \hat{u}(b; s) - \beta = 0$$

2.4 Homotopy Perturbation Method (HPM)

HPM constructs a homotopy between an initial guess solution and the exact solution [15]. Consider:

$$\mathcal{L}[u(x)] + \mathcal{N}[u(x)] = f(x)$$

where $\mathcal{L}$ is a linear operator and $\mathcal{N}$ a non-linear operator. The homotopy is defined as:

$$H(v, p) = (1 - p)[\mathcal{L}(v) - \mathcal{L}(u_0)] + p[\mathcal{L}(v) + \mathcal{N}(v) - f(x)] = 0$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation. The solution is expressed as a series in p:

$$v(x) = v_0(x) + p\, v_1(x) + p^2\, v_2(x) + \cdots$$

As $p \to 1$, the approximate solution converges to the solution of the original problem.

3. ERROR ANALYSIS

The performance of any numerical method is based on how close the approximate solution is to the exact solution of the BVP. In the use of the Galerkin method, modified Galerkin method, Shooting method, and HPM, there are two basic sources of error that need to be taken into consideration: truncation error and round-off error. A truncation error occurs when we stop after a limited number of steps.

The Shooting method shows truncation error because of using the step-by-step method, such as Runge-Kutta integration, which only gives approximate results [4]. Truncation error is associated with the Galerkin and modified Galerkin methods, which only use a finite-dimensional trial space, thus eliminating components that are not included in the selected basis. Truncation in the Homotopy Perturbation method, after a finite number of approximations, stops the perturbation series.

Round-off error is an artifact of finite precision arithmetic that runs on computers. Because real numbers are represented using a limited number of digits, repeated operations may produce a minor difference between the actual and computed values. Individually, the errors happen to be quite negligible, but when combined, they can become serious. Overall, numerical error, truncation, and round-off effects are added together to give the total numerical error in practice. In Galerkin-type methods, truncation is more dominant as a result of projection to a smaller functional space. By contrast, more accurate iterative algorithms like the Shooting method and HPM are less robust to the round-off error when high accuracy is needed or a large number of iterations must be performed. Therefore, step size, selection of basis functions, and truncation levels must be strictly controlled to achieve compromises between computational cost and numerical accuracy.

4. ALGORITHMS FOR SOLVING BOUNDARY VALUE

4.1 Galerkin Method

The following is a description and presentation of the Galerkin method algorithm [8]:

i. Define the operator:

$$\mathcal{L}[u(x)] = f(x)$$

ii. Select basis functions: Choose trial functions $\phi_i(x)$.

iii. Approximate the solution:

$$u(x) \approx \Sigma\, c_i \phi_i(x)$$

iv. Form the residual:

$$R(x) = \mathcal{L}[u(x)] - f(x)$$

v. Apply Galerkin condition:

$$\int_a^b R(x)\varphi_j\,(x)dx = 0, j = 1, \dots, n$$

vi. Construct a system of equations:

$$Kc = F$$

$$K_{ij} = \int_a^b (\mathcal{L}[\varphi_i(x)])\,\varphi_j(x)dx$$

$$F_j = \int_a^b f(x)\,\varphi_j(x)dx$$

vii. Solve for coefficients and find $\{c_i\}$.

viii. Regenerate solution:

$$u(x) = \Sigma\, c_i \phi_i(x)$$

4.2 Modified Galerkin Method

The following is an outline and formulation of the Galerkin method's process [12]:

i. Define the operator:

$$\mathcal{L}[u(x)] = f(x)$$

ii. Choose modified basis functions: $\psi_i(x)$.

iii. Approximate the solution:

$$u(x) \approx \Sigma\, c_i \psi_i(x)$$

iv. Form residual:

$$R(x) = \mathcal{L}[u(x)] - f(x)$$

v. Select weight functions: $w_j(x)$.

vi. Apply weighted residual condition:

$$\int_a^b R(x)\, w_j(x) dx = 0, j = 1, \ldots, n$$

vii. Form an algebraic system:

$$Kc = F$$

viii. Solve for coefficients and obtain $\{c_i\}$.

ix. Reconstruct solution:

$$u(x) = \Sigma\, c_i \psi_i(x)$$

4.3 Shooting Method

The algorithm for the Shooting method is as follows [4]:

i. Define the problem:

$$u''(x) = g(x, u, u'),\ u(a) = \alpha,\ u(b) = \beta$$

ii. Convert to IVP:

$$u(a) = \alpha,\ u'(a) = s$$

iii. Choose initial guesses:

iv. Start with two trial slopes s_0, s_1, step size h, and tolerance ε.

v. Iterative process:
  a) Solve the IVP using RK4.
  b) Compute the difference at the end point:

$$\varphi(s) = \hat{u}(b;\, s) - \beta$$

  c) If $|\varphi(s)| < \varepsilon$, then boundary condition satisfied.
  d) Otherwise, update the slope using the Secant (or Newton) method:

$$s_{k+1} = s_k - \frac{\varphi(s_k)\,(s_k - s_{k-1})}{(\varphi(s_k) - \varphi(s_{k-1}))}$$

vi. Result:
The corrected slope gives the approximate solution $u(x)$ that satisfies both boundary conditions.

4.4 Homotopy Perturbation Method (HPM)

The algorithm can be described as follows [16]:

i. Define the problem:

$$\mathcal{L}[u(x)] + \mathcal{N}[u(x)] = f(x)$$

ii. Choose an initial guess:

$$u_0(x)$$

iii. Construct the homotopy:

$$H(v,p) = (1-p)[\mathcal{L}(v) - \mathcal{L}(u_0)] + p[\mathcal{L}(v) + N(v) - f(x)] = 0$$

iv. Assume a series expansion:

$$v(x) = v_0(x) + pv_1(x) + p^2 v_2(x) + \cdots$$

Substitute into homotopy and collect terms in powers of p.

Equating coefficients:

$$p^0: v_0(x)$$
$$p^1: v_1(x)$$

$$p^2: v_2(x)$$

Solve sequentially for $v_0, v_1, v_2, ...$

v. Find the approximate solution:

$$u(x) \approx v_0(x) + v_1(x) + v_2(x) + \cdots$$

vi. Check the convergence of the series.

5. RESULT DISCUSSION

The Galerkin method, the modified Galerkin method, the Shooting method, and the Homotopy Perturbation method were used to obtain numerical solutions to different BVPs. These numerical solutions have been compared with the analytical solutions to find out the errors. Analysis of the RMS errors gives a clear idea of the correctness and dependability of every technique. All the problems have been taken from Numerical Analysis by Burden et al. [25].

Problem 1. The BVPs

$$y'' = 4(y - x), 0 \le x \le 1, y(0) = 0, y(1) = 2$$

has the analytical solution is $y(x) = e^2(e^4 - 1)^{-1}(e^{2x} - e^{-2x}) + x,$ and $h = 0.1.$

Four numerical methods the Galerkin method, and modified Galerkin method, Shooting method, and Homotopy Perturbation method were used to get the numerical results in Table 1. To determine comparative accuracy, each solution is compared to the analytical solution at discrete locations x_i between 0.0 and 1.0 with a step size of 0.1.

Table 1. Comparison of numerical solution and errors from problem 1.

x_i	Exact Solution	Galerkin Method	Error (Galerkin Method)	Modified Galerkin Method	Error (Modified Galerkin Method)	Shooting Method	Error (Shooting Method)	HPM	Error (HPM)
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.155512	0.155414	0.000099	0.155610	0.000098	0.155517	0.000004	0.120033	0.035479
0.2	0.313253	0.313059	0.000194	0.313445	0.000192	0.313287	0.000034	0.244106	0.069147
0.3	0.475538	0.475257	0.000281	0.475817	0.000279	0.475564	0.000026	0.376421	0.099118
0.4	0.644869	0.644513	0.000357	0.645223	0.000354	0.644864	0.000005	0.521503	0.123366
0.5	0.824027	0.823614	0.000413	0.824437	0.000410	0.824015	0.000012	0.684375	0.139652
0.6	1.016190	1.015747	0.000443	1.016628	0.000439	1.016232	0.000043	0.870736	0.145453
0.7	1.225055	1.224621	0.000434	1.225485	0.000430	1.225091	0.000036	1.087154	0.137901
0.8	1.454993	1.454622	0.000371	1.455361	0.000368	1.454961	0.000032	1.341276	0.113717
0.9	1.711218	1.710982	0.000236	1.711451	0.000233	1.711169	0.000049	1.642055	0.069163
1.0	2.000000	2.000000	0.000000	2.000000	0.000000	2.000000	0.000000	2.000000	0.000000

Figure 1 illustrates the graphical values of the solutions obtained by the four methods for problem 1:

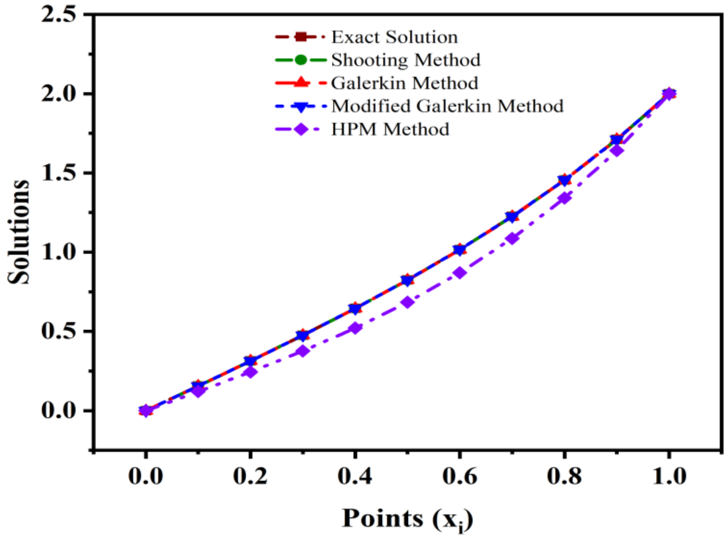

Figure 1. Plot of the computed numerical solutions for problem 1.

Figure 2 shows the errors gathered from the numerical values:

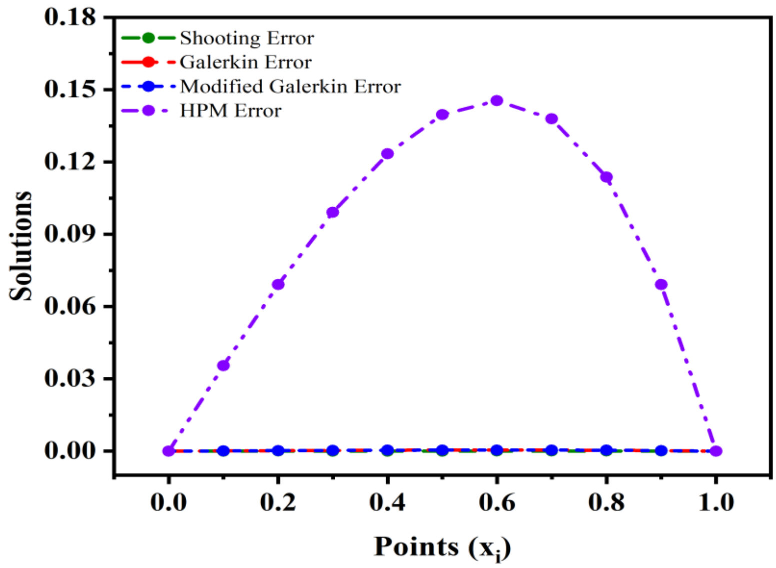

Figure 2. Error analysis plot corresponding to problem 1.

Among the four methods studied, the Shooting method was the most accurate, yielding the lowest root mean square (RMS) error of 2.79×10^{-5}. Its high accuracy is caused by its approach of converting a boundary value problem into an initial value problem, which is then solved effectively using numerical integration. The error for this method was consistently low across the entire domain, indicating its solutions closely approximate the analytical values. In contrast, the Galerkin and modified Galerkin methods had lower accuracy, with RMS errors of 3.02×10^{-4} and 2.99×10^{-4}, respectively. The standard and modified Galerkin schemes are useful and efficient despite being slightly less accurate, as the derived errors are fairly consistent throughout the computation domain. The Homotopy Perturbation method (HPM) was the least accurate, demonstrating the largest deviation from the exact solution with an RMS error of 9.93×10^{-2}.

Problem 2. The BVPs

$$y'' = y' + 2(y - lnx)^3 - x^{-1}, 2 \leq x \leq 3, y(2) = \frac{1}{2} + ln2, y(3) = \frac{1}{3} + ln3$$

has the analytical solution is $y(x) = x^{-1} + lnx$, and $h = 0.1$.

Table 2 presents a comparison of four methods, as well as their errors, where the step size is $h = 0.1$ and x_i is taken from 2.0 to 3.0.

Table 2. Comparison of numerical solution and errors from problem 2.

x_i	Exact Solution	Galerkin Method	Error (Galerkin Method)	Modified Galerkin	Error (Modified Galerkin Method)	Shooting Method	Error (Shooting Method)	HPM	Error (HPM)
2.0	1.193147	1.193147	0.000000	1.193147	0.000000	1.193147	0.000000	1.193147	0.000000
2.1	1.218128	1.197296	0.020832	1.213087	0.005041	1.218039	0.000089	1.196496	0.021632
2.2	1.243003	1.204629	0.038374	1.230725	0.012278	1.242930	0.000073	1.203081	0.039922
2.3	1.267692	1.215562	0.052130	1.247083	0.020609	1.267559	0.000133	1.213403	0.054288
2.4	1.292135	1.230522	0.061614	1.263957	0.028178	1.291925	0.000210	1.227949	0.064187
2.5	1.316291	1.249951	0.066340	1.285113	0.031178	1.316292	0.000001	1.247198	0.069092
2.6	1.340127	1.274320	0.065807	1.309165	0.030962	1.339870	0.000257	1.271642	0.068485
2.7	1.363622	1.304133	0.059489	1.338274	0.025348	1.363449	0.000173	1.301787	0.061835
2.8	1.386762	1.339937	0.046825	1.370209	0.016553	1.386580	0.000183	1.338169	0.048593
2.9	1.409538	1.382325	0.027213	1.402624	0.006914	1.409263	0.000276	1.381354	0.028185
3.0	1.431946	1.431946	0.000000	1.431946	0.000000	1.431946	0.000000	1.431946	0.000000

The graphical values of the four approaches for problem 2 are shown in Figure 3:

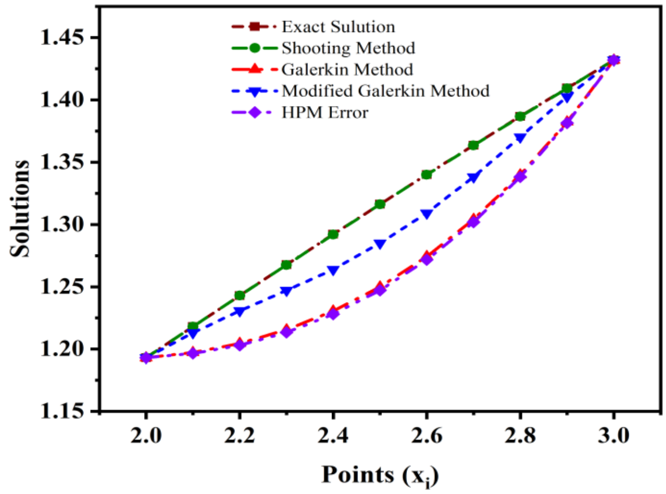

Figure 3. Plot of the computed numerical solutions for problem 2.

The errors derived from the numerical values of problem 2 are presented in Figure 4:

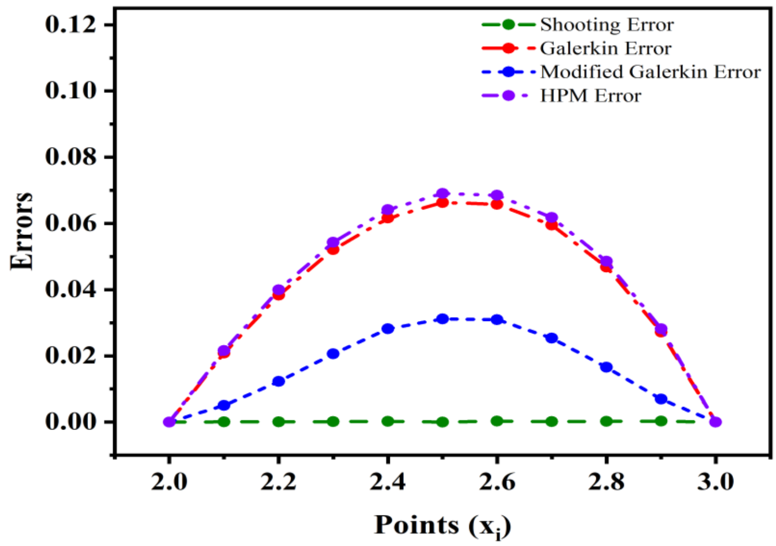

Figure 4. Error analysis plot corresponding to problem 2.

To determine the performance of each method, the root mean square (RMS) error was calculated. The Shooting method had the minimum RMS error of 1.59×10^{-4}, which means that it is most accurate and it is in close proximity to the actual solution. The RMS error of the modified Galerkin method was 1.97×10^{-2}, which is quite moderate and ranks this method as the second most accurate. Even though the deviations in this approach were somewhat larger, it nevertheless showed quite reasonable consistency with the analytical solution during the analyzed period. The standard Galerkin method displays the greater errors in the interior of the domain, giving it a root mean square error of 4.63×10^{-2}; the method is therefore not as accurate as the Shooting method or the modified Galerkin scheme. The weakest among the four methods is the Homotopy Perturbation method, with the largest RMS error of 4.82×10^{-2}, and its solution showing large deviations from the analytic solution.

Problem 3. The BVP

$$y'' = -e^{-2y}, 1 \leq x \leq 2, y(1) = 0, y(2) = ln2$$

has the analytical solution is $y(x) = lnx$, and $h = 0.1$.

The comparison of the four methods in terms of their numbers is in Table 3 under Galerkin, modified Galerkin, Shooting, and Homotopy Perturbation method, where the step size is $\mathrm{h} = 0.1$ and $\mathrm{x_i}$ is taken from 1.0 to 2.0.

Table 3. Comparison of numerical solution and errors from the problem.

x_i	Exact Solution	Galerkin Method	Error (Galerkin Method)	Modified Galerkin Method	Error (Modified Galerkin Method)	Shooting Method	Error (Shooting Method)	HPM	Error (HPM)
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1.1	0.095310	0.095386	0.000076	0.095234	0.000076	0.095308	0.000002	0.094870	0.000440
1.2	0.182322	0.182440	0.000118	0.182203	0.000119	0.182325	0.000003	0.181527	0.000795
1.3	0.262364	0.262504	0.000140	0.262226	0.000138	0.262359	0.000005	0.261331	0.001033
1.4	0.336472	0.336616	0.000144	0.336329	0.000143	0.336450	0.000022	0.335328	0.001144
1.5	0.405465	0.405601	0.000136	0.405330	0.000135	0.405441	0.000024	0.404330	0.001135
1.6	0.470003	0.470123	0.000120	0.469884	0.000119	0.469988	0.000015	0.468983	0.001020
1.7	0.530628	0.530725	0.000097	0.530531	0.000097	0.530618	0.000010	0.529801	0.000827
1.8	0.587787	0.587855	0.000068	0.587718	0.000069	0.587779	0.000008	0.587208	0.000579
1.9	0.641854	0.641890	0.000036	0.641818	0.000036	0.641849	0.000005	0.641557	0.000297
2.0	0.693147	0.693147	0.000000	0.693147	0.000000	0.693147	0.000000	0.693147	0.000000

Figure 5 displays the graphical values of the four methods for the problem 3:

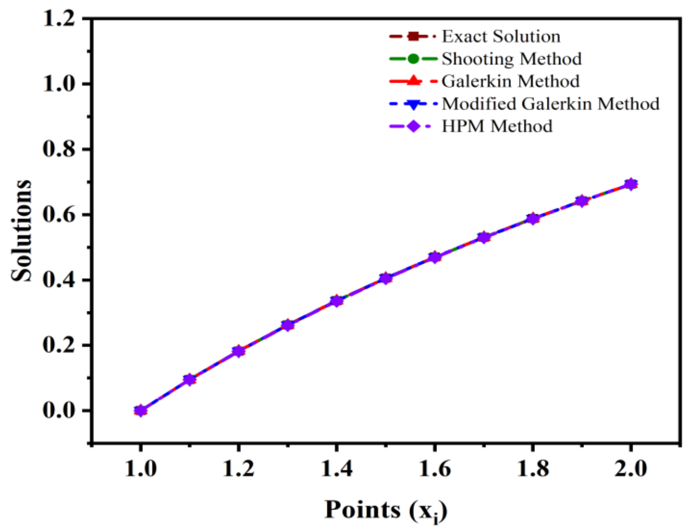

Figure 5. Plot of the computed numerical solutions for problem 3.

Figure 6 displays the errors obtained from the numerical values of problem 3:

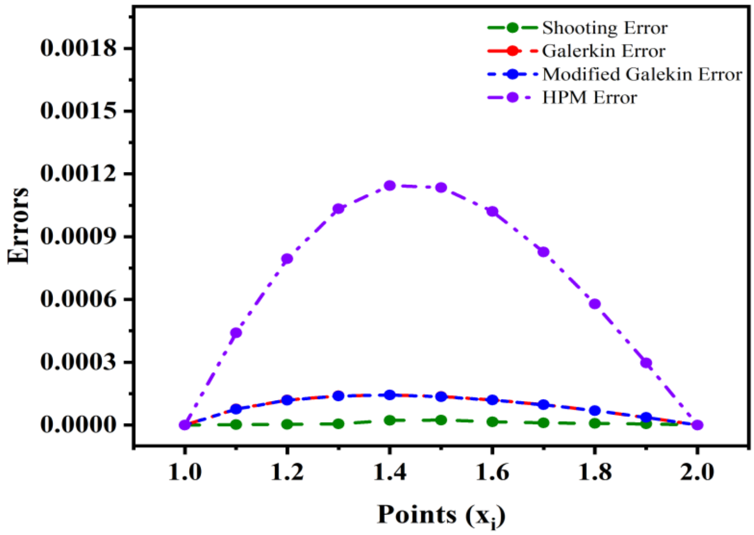

Figure 6. Error analysis plot corresponding to problem 3.

Among the numerical methods evaluated for solving the BVP in the interval $1 \leq x \leq 2$ with a step size of $h = 0.1$, the Shooting method proved to be the most accurate, exhibiting the lowest root mean square error of approximately 1.17×10^{-5}. The modified Galerkin method followed closely with an RMS error of 9.88×10^{-5}, demonstrating good agreement with the analytical solution. The standard Galerkin method was less precise, with an RMS error of about 9.92×10^{-5}, indicating larger deviations. A maximum deviation by the HPM occurs with an RMS error of 7.77×10^{-4}. In general, the Shooting method offers the best solution with the smallest errors of all points. The modified Galerkin method offers a good trade-off between accuracy and computational performance, which makes it a viable option for a variety of problems. The standard Galerkin method, although slightly less accurate than its modified variant, nevertheless yields quite stable outcomes. On the other hand, HPM, even if inaccurate, can at least be useful in cases of quickly acquiring rough estimates where a limited number of computational resources are available.

Problem 4. The BVPs

$$y'' = 2y^3 - 6y - 2x^3, 1 \leq x \leq 2, y(1) = 2, y(2) = \frac{5}{2}$$

has the solution $y(x) = x + x^{-1}$, where $h = 0.1$.

Comparative analysis of four numerical techniques, which are the Galerkin method, modified Galerkin numerical method, the Shooting method, and Homotopy Perturbation method reveals divergent deviations of different techniques from the exact solution in the interval $1 \leq x \leq 2$ with a step size of $h = 0.1$ as shown in Table 4.

Table 4. Comparison of numerical solution and errors from problem 4.

x_i	Exact Solution	Galerkin Method	Error (Galerkin Method)	Modified Galerkin	Error (Modified Galerkin Method)	Shooting Method	Error (Shooting Method)	HPM	Error (HPM)
1.0	2.000000	2.000000	0.000000	2.000000	0.000000	2.000000	0.000000	2.000000	0.000000
1.1	2.009090	2.008772	0.000318	2.008949	0.000141	2.009087	0.000003	2.006000	0.003090
1.2	2.033333	2.032876	0.000457	2.033151	0.000182	2.033327	0.000006	2.027455	0.005878
1.3	2.069230	2.068735	0.000495	2.069053	0.000177	2.069220	0.000010	2.061140	0.008090

1.4	2.114285	2.113807	0.000478	2.114132	0.000153	2.114269	0.000016	2.104774	0.009511
1.5	2.166667	2.166234	0.000433	2.166544	0.000123	2.166642	0.000025	2.156667	0.010000
1.6	2.225000	2.224627	0.000373	2.224906	0.000094	2.224959	0.000011	2.215489	0.009511
1.7	2.288235	2.287930	0.000305	2.288167	0.000068	2.288169	0.000006	2.280145	0.008090
1.8	2.355556	2.355329	0.000227	2.355511	0.000045	2.355553	0.000003	2.349678	0.005878
1.9	2.426316	2.426185	0.000131	2.426283	0.000033	2.426315	0.000001	2.423226	0.003090
2.0	2.500000	2.500000	0.000000	2.500000	0.000000	2.500000	0.000000	2.500000	0.000000

The graphical values of the four approaches for problem 4 are shown in Figure 7:

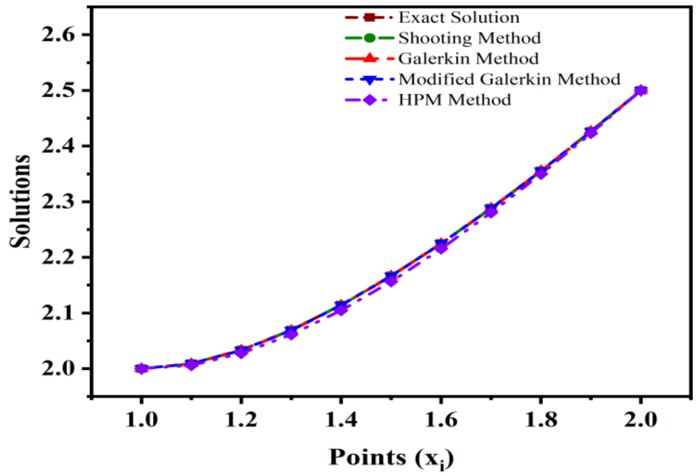

Figure 7. Plot of the computed numerical solutions for problem 4.

The errors derived from the numerical values in problem 4 are shown in Figure 8:

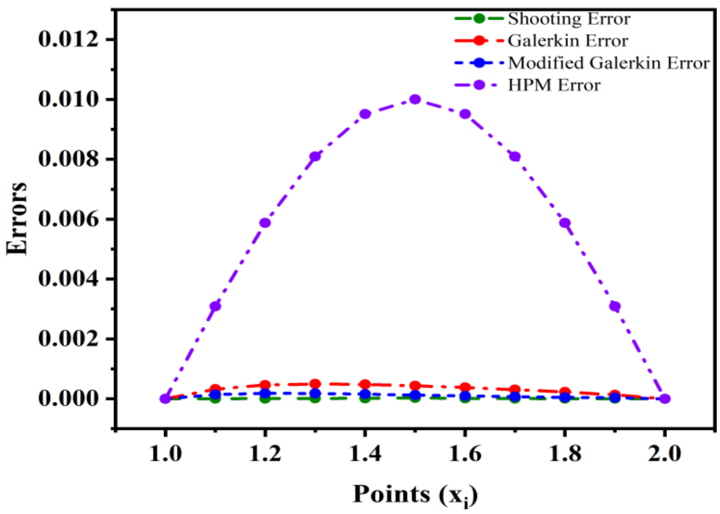

Figure 8. Error analysis plot corresponding to problem 4.

The Shooting method is the most accurate of the four methods for this problem, since it gives the numerical solutions that follow the exact solution closely in the sense that the RMS error is at 1.04×10^{-5}. The Modified Galerkin method achieves an RMS error of 1.13×10^{-4}, which gives lower accuracy than the Shooting method. An RMS error of 3.40×10^{-4} is registered by the standard Galerkin method that gives better accuracy than the Homotopy Perturbation method. Although both the Galerkin method and the modified Galerkin method tend to follow the direction of the exact solution, the modified Galerkin method provides more sophisticated results. Conversely, the Homotopy Perturbation method provided an RMS error of 6.74×10^{-3}, which is higher than all of three methods. Such results suggest that the HPM is relatively less effective at resolving details of the solution at fine scales when compared to the other three numerical solutions.

6. LIMITATIONS

The findings of this study are significant and applicable, but some shortcomings must be admitted. In this paper, relatively simple problems have been considered, where the solutions are not very complicated. As a result, the findings of this study work well for that category of problems. But when they get more complicated, e.g., when they have many variables or dimensions, when they are highly nonlinear, or when they involve especially difficult equations, then the schemes used in the present paper may behave quite differently; in that case, the results of this paper might not be completely relevant. Galerkin trial and bases functions were pre-determined, and there may be substantial changes in quantities when they are changed differently. Similarly, in the case of HPM, the homotopy and guessing building were selected in optimal conditions, but in reality, they can have a significant effect on convergence. Other numerical strategies are not included in the comparison, such as finite difference, finite element, and spectral methods, which may help enhance the understanding of the strengths and weaknesses of the four methods discussed. Additionally, the study does not take into account the numerical round-off errors, sensitivity of the parameters, and noisy input data, which are common in practice. Therefore, due to the low number of issues examined, the findings cannot be considered exhaustive, and the analysis of a larger range of issues would strengthen the viability and universality of the findings.

7. CONCLUSION

A comparative analysis of four methods, namely the Galerkin method, the Modified Galerkin method, the Shooting method, and the Homotopy Perturbation method, is also provided in this paper. The results suggest that the Shooting technique gives maximum accuracy in each of the tested cases and is very close to the analytic solutions. The modified Galerkin method is also very good, as it has minimal deviations and is slightly better than the standard Galerkin method. Although HPM is effective, it shows slightly larger deviations compared to the other methods.

ACKNOWLEDGEMENTS

The authors sincerely acknowledge that the problems addressed in this paper were derived from Richard L. Burden's book, Numerical Analysis, 10th edition.

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