International Journal of Mathematics, Statistics, and Computer Science
Vol. 1, 2023
ISSN: 2704-1077, DOI: https://doi.org/10.59543/ijmscs.v1i.7737

On The Product and Ratio of Pareto and Erlang Random Variables

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Received 10.03.2023, Revised 20.04.2023, Accepted 29.04.2023, Published 09.05.2023

ABSTRACT

The distributions of products and ratios of random variables are of interest in many areas of the sciences. In this paper, we find analytically the probability distributions of the product XY and the ratio X/Y , when X and Y are two independent random variables following Pareto and Erlang distributions, respectively.
Keywords: Product Distribution, Ratio Distribution, Pareto Distribution, Erlang Distribution, Exponential Distribution, probability density function, Moment of order r, Survival function, Hazard function.

1. INTRODUCTION

Engineering, Physics, Economics, Order statistics, Classification, Ranking, Selection, Number theory, Genetics, Biology, Medicine, Hydrology, Psychology, these all applied problems depend on the distribution of product and ratio of random variables [1],[2].
As an example of the use of the product of random variables in physics, Sornette [27] mentions: To mimic system size limitation, Takayasu, Sato, and Takayasu introduced a threshold x_c ... and found a stretched exponential truncating the power-law pdf beyond x_c. Frisch and Sornette recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to the multiplication of random variables, predicts the generic presence of stretched exponential pdfs. The problem thus boils down to determining the tail of the pdf for a product of random variables. Several authors have studied the product distributions for independent random variables that come from the same family or different families, see [21] for t and Rayleigh families, [4] for Pareto and Kumaraswamy families, [6] for the t and Bessel families, and [22] for the independent generalized Gamma-ratio family, [28] for Pareto and Rayleigh families. As an example of the use of the ratio of random variables include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, and inventory ratios in economics. Several authors have studied the ratio distributions for independent random variables come from the same family or different families. the historical review, see [9], [10] for the Normal family, [11] for Student's t family, [12] for the Weibull family, [13] for the noncentral Chi-squared family, [14] for the Gamma family, [15] for the Beta family, [16] for the Logistic family, [17] for the Frechet family, [3] for the inverted Gamma family, [18] for Laplace family, [7] for the generalized-F family, [19] for the Hypoexponential family, [2] for the Gamma and Rayleigh families, and [20] for Gamma and Exponential families, [28] for Pareto and Rayleigh families.
In this paper, we find analytically the probability distributions of the product XY and the ratio X/Y , when X and Y are two independent random variables following Pareto and Erlang distributions respectively. with probability density functions (p.d.f.s)

$$f_X(x) = \frac{ca^c}{x^{c+1}} \tag{1}$$

$$f_Y(y) = \frac{y^{\alpha-1}e^{-\frac{y}{\lambda}}}{\lambda^\alpha\Gamma(\alpha)} \tag{2}$$

respectively, for $a \leq x < \infty$, $a > 0$, $c > 0$, $y \geq 0$, $\lambda > 0$, $\alpha > 0$, α is an integer for the Erlang distribution.

NOTATIONS AND PRELIMINARIES

Recall some special mathematical functions, these will be used repeatedly throughout this article.

- The upper incomplete gamma function defined by

$$\Gamma(a,x) = \int_x^{\infty} \exp(-t)t^{a-1}dt \tag{3}$$

- The lower incomplete gamma function defined by

$$\gamma(a,x) = \int_0^{x} \exp(-t)t^{a-1}dt \tag{4}$$

The calculations of this paper involve several Lemmas

Lemma 1 For $\alpha \geq 0$, $r \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$

$$I(\alpha, r, b) = \int_{\alpha}^{\infty} x^r e^{-bx} dx = \frac{1}{b^{r+1}} \Gamma(r+1, b\alpha) \tag{5}$$

Proof Let $u = bx$, then

$$I(\alpha, r, b) = \int_{b\alpha}^{+\infty} \frac{u^r}{b^{r+1}} e^{-u} du = \frac{1}{b^{r+1}} \Gamma(r+1, b\alpha) \tag{6}$$

Lemma 2 For $t \in \mathbb{R}$,

$$\frac{d}{dx}\Gamma(t, v(x)) = -v(x)^{t-1} e^{-v(x)} \frac{d}{dx} v(x) \tag{7}$$

Proof

$$\frac{d}{dx}\Gamma(t, v) = \frac{d}{dv}\Gamma(t, v)\frac{dv}{dx} \tag{8}$$

Note that

$$\frac{d}{dv}\Gamma(t, v) = -v^{t-1}e^{-v} \tag{9}$$

Lemma 3 For $\alpha \geq 0$, $r \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$

$$\int_0^{\alpha} x^r e^{-bx} dx = \frac{1}{b^{r+1}} \gamma(r+1, b\alpha) \tag{10}$$

Proof For $u = bx$

$$\int_0^{b\alpha} \frac{u^r}{b^{r+1}} e^{-u} du = \frac{1}{b^{r+1}} \gamma(r+1, b\alpha) \tag{11}$$

2. DISTRIBUTION OF THE RATIO X/Y

Theorem 2.1. Suppose X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ the cumulative distribution function *c.d.f.* of $Z = X/Y$ can be expressed as:

$$F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{\Gamma(\alpha, \frac{a}{\lambda z})}{\Gamma(\alpha)} - \frac{a^c}{\Gamma(\alpha) z^c \lambda^c} \Gamma(\alpha - c, \frac{a}{\lambda z}) & \text{if } z > 0 \end{cases} \tag{12}$$

Proof. the c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$ Thus, one can write the c.d.f. of X/Y as

$$\begin{aligned} Pr(X/Y \leq z) &= \int_{a/z}^{\infty} F_X(zy) f_Y(y) dy \\ &= \int_{a/z}^{\infty} (1 - \frac{a^c}{z^c y^c}) f_Y(y) dy \\ &= \int_{a/z}^{\infty} f_Y(y) dy - \frac{a^c}{z^c} \int_{a/z}^{\infty} \frac{y^{\alpha-1} e^{-\frac{y}{\lambda}}}{y^c \lambda^\alpha \Gamma(\alpha)} dy \end{aligned} \tag{13}$$

Let

$$I = \int_{a/z}^{\infty} f_Y(y) dy = \int_{0}^{\infty} f_Y(y) dy - \int_{0}^{a/z} f_Y(y) dy = 1 - \int_{0}^{a/z} f_Y(y) dy \tag{14}$$

If we substitute $u = \frac{y}{\lambda}$ in the integral above we obtain

$$I = \frac{\Gamma(\alpha, \frac{a}{z\lambda})}{\Gamma(\alpha)} \tag{15}$$

And let

$$J = \int_{a/z}^{\infty} \frac{a^c y^{\alpha-1} e^{-\frac{y}{\lambda}}}{z^c y^c \lambda^\alpha \Gamma(\alpha)} dy \tag{16}$$

If we substitute $u = \frac{y}{\lambda}$, we obtain

$$J = \frac{a^c}{\Gamma(\alpha) z^c \lambda^\alpha} \lambda^{\alpha - c} \Gamma(\alpha - c, \frac{a}{\lambda z}) \tag{17}$$

Finally we get

$$F_Z(z) = \frac{\Gamma(\alpha, \frac{a}{\lambda z})}{\Gamma(\alpha)} - \frac{a^c}{\Gamma(\alpha) z^c \lambda^c} \Gamma(\alpha - c, \frac{a}{\lambda z}) \tag{18}$$

Corollary 2.2. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ the probability density function *p.d.f.* of $Z = X/Y$ can be expressed as:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{c a^c \Gamma(\alpha - c, \frac{a}{\lambda z})}{\Gamma(\alpha) \lambda^c z^{c+1}} & \text{if } z > 0 \end{cases} \tag{19}$$

Proof. The probability density function $f_Z(z)$ in (19) easily follows by differentiation and by using **Lemma 2**.

Corollary 2.3. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > r$. the moment of order r of $Z = X/Y$ can be expressed as:

$$E[Z^r] = \frac{c a^c \beta^{r-c} \Gamma(\alpha - c, \frac{a}{\lambda \beta})}{\Gamma(\alpha) \lambda^c (c - r)} + \frac{c a^r \left[\Gamma(c - r) - \Gamma(c - r, \frac{a}{\beta \lambda})\right]}{\Gamma(\alpha) \lambda^r (c - r)} \tag{20}$$

Proof.

$$\begin{aligned} E[Z^r] &= \int_{-\infty}^{+\infty} z^r f_Z(z)dz \\ &= \int_{\beta}^{\infty} z^r \frac{ca^c\Gamma(\alpha-c,\frac{a}{\lambda z})}{\Gamma(\alpha)\lambda^c z^{c+1}}dz \end{aligned} \tag{21}$$

Integration by parts implies:

$$E[Z^r] = \frac{ca^c}{\Gamma(\alpha)\lambda^c}\left[\Gamma(\alpha-c,\frac{a}{\lambda\beta})\frac{\beta^{r-c}}{(c-r)} - \int_{\beta}^{\infty}\frac{e^{-a/\lambda z}a^{\alpha-c}z^{r-c}}{\lambda^{\alpha-c}z^{\alpha-c+1}(r-c)}dz\right] \tag{22}$$

If we substitute $u=\frac{a}{\lambda z}$ and using **Lemma 3** in the integral above we obtain

$$E[Z^r] = \frac{ca^c\beta^{r-c}\Gamma(\alpha-c,\frac{a}{\lambda\beta})}{\Gamma(\alpha)\lambda^c(c-r)} + \frac{ca^r\Big[\Gamma(c-r)-\Gamma(c-r,\frac{a}{\beta\lambda})\Big]}{\Gamma(\alpha)\lambda^r(c-r)} \tag{23}$$

Corollary 2.4. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c>1.$ the Expected value of $Z=X/Y$ can be expressed as:

for $r=1$

$$E[Z] = \frac{ca^c\beta^{1-c}\Gamma(\alpha-c,\frac{a}{\lambda\beta})}{\Gamma(\alpha)\lambda^c(c-1)} + \frac{ca\Big[\Gamma(c-1)-\Gamma(c-1,\frac{a}{\beta\lambda})\Big]}{\Gamma(\alpha)\lambda(c-r)} \tag{24}$$

Corollary 2.5. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c>2$ the Variance of $Z=X/Y$ can be expressed as:

$$\begin{aligned} \sigma^2 = &\left[\frac{ca^c\beta^{2-c}\Gamma(\alpha-c,\frac{a}{\lambda\beta})}{\Gamma(\alpha)\lambda^c(c-2)} + \frac{ca^2\Big[\Gamma(c-2)-\Gamma(c-2,\frac{a}{\beta\lambda})\Big]}{\Gamma(\alpha)\lambda^2(c-2)}\right] \\ &-\left[\frac{ca^c\beta^{1-c}\Gamma(\alpha-1,\frac{a}{\lambda\beta})}{\Gamma(\alpha)\lambda^c(c-1)} + \frac{ca\Big[\Gamma(c-1)-\Gamma(c-1,\frac{a}{\beta\lambda})\Big]}{\Gamma(\alpha)\lambda(c-1)}\right]^2 \end{aligned} \tag{25}$$

Proof. By definition the variance of X/Y is:

$$\sigma^2 = E[Z^2]-E[Z]^2 \tag{26}$$

Corollary 2.6. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z>0$ the Survival function of $Z=X/Y$ can be expressed as:

$$S_Z(z) = \begin{cases} 1 & \text{if } z\le 0 \\ 1-\frac{\Gamma(\alpha,\frac{a}{\lambda z})}{\Gamma(\alpha)} + \frac{a^c}{\Gamma(\alpha)z^c\lambda^c}\Gamma(\alpha-c,\frac{a}{\lambda z}) & \text{if } z>0 \end{cases} \tag{27}$$

Proof By definition of the survival function

$$S_Z(z) = 1-F_Z(z). \tag{28}$$

Corollary 2.7. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ the Hazard function of $Z = X/Y$ can be expressed as

$$h_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{ca^c\Gamma(\alpha-c,\frac{a}{\lambda z})}{\Gamma(\alpha)z^{c+1}\lambda^c - z^{c+1}\lambda^c\Gamma(\alpha,\frac{a}{\lambda z}) + a^c z\Gamma(\alpha-c,\frac{a}{\lambda z})} & \text{if } z > 0 \end{cases} \tag{29}$$

Proof By definition of the hazard function

$$h_Z(z) = \frac{f_Z(z)}{S_Z(z)}. \tag{30}$$

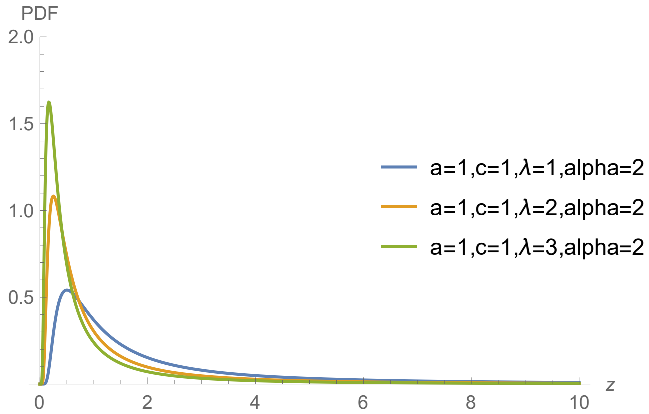

Figure 1. Plots of the pdf (19) for $a = 1, c = 1, \lambda = 1, 2, 3, \alpha = 2$

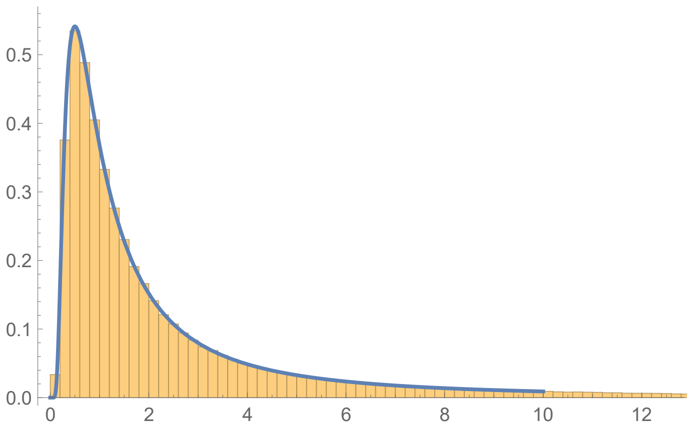

Figure 2. Monte carlo simulation for the ratio of pareto and erlang, for $a = 1, c = 1, \lambda = 1, \alpha = 2$

2.1. Distribution of the ratio of Pareto and Exponential random variables

The Erlang distribution is the distribution of a sum of α independent Exponential variables with mean $\frac{1}{b} = \lambda$ each. When $\alpha = 1$ the Erlang distribution Y simplifies to the Exponential distribution witch lead us to the following corollary.

Corollary 2.8. Suppose X and Y are two independent Pareto (1) and Erlang (2) random variables, then for $z > 0$, $\alpha = 1$. The probability density function p.d.f. of the ratio of Pareto and Exponential random variables $Z = X/Y$ can be expressed as:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{c(ab)^c \Gamma(1-c, \frac{ab}{z})}{z^{c+1}} & \text{if } z > 0 \end{cases} \tag{31}$$

3. DISTRIBUTION OF THE PRODUCT XY

Theorem 3.1. Suppose X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ The cumulative distribution function *c.d.f.* of $Z = XY$ can be expressed as:

$$F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ 1 - \frac{\Gamma(\alpha, z/a\lambda)}{\Gamma(\alpha)} - \frac{a^c \lambda^c \Gamma(\alpha+c)}{z^c \Gamma(\alpha)} + \frac{a^c \lambda^c \Gamma(\alpha+c, z/a\lambda)}{z^c \Gamma(\alpha)} & \text{if } z > 0 \end{cases} \tag{32}$$

Proof. the c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$ Thus, one can write the c.d.f. of XY as

$$\begin{aligned} F_Z(z) = Pr(Z \le z) = Pr(XY \le z) &= \int_0^\infty F_X(z/y) f_Y(y) dy \\ &= \int_0^{z/a} (1 - \frac{a^c y^c}{z^c}) f_Y dy \\ &= \int_0^{z/a} f_Y(y) dy - \int_0^{z/a} \frac{a^c y^c}{z^c} f_Y(y) dy \end{aligned} \tag{33}$$

Let

$$\begin{aligned} I &= \int_0^{z/a} f_Y(y) dy \\ &= \int_0^\infty f_Y(y) dy - \int_{z/a}^\infty f_Y(y) dy \\ &= 1 - \int_{z/a}^\infty f_Y(y) dy \end{aligned} \tag{34}$$

Using **Lemma 1** in the integral above we get

$$I = 1 - \frac{\Gamma(\alpha, z/a\lambda)}{\Gamma(\alpha)} \tag{35}$$

Now Let

$$\begin{aligned} J &= \int_0^{z/a} \frac{a^c y^c}{z^c} f_Y(y) dy \\ &= \frac{a^c}{z^c \lambda^\alpha \Gamma(\alpha)} \int_0^{z/a} y^{\alpha+c-1} e^{-y/\lambda} dy \end{aligned} \tag{36}$$

Using **Lemma 3** in the integral above we get

$$J = \frac{a^c \lambda^c}{z^c \Gamma(\alpha)} \Big[\Gamma(\alpha + c) - \Gamma(\alpha + c, z/a\lambda)\Big]$$

Finally

$$F_Z(z) = I - J = 1 - \frac{\Gamma(\alpha, z/a\lambda)}{\Gamma(\alpha)} - \frac{a^c \lambda^c \Gamma(\alpha + c)}{z^c \Gamma(\alpha)} + \frac{a^c \lambda^c \Gamma(\alpha + c, z/a\lambda)}{z^c \Gamma(\alpha)} \tag{37}$$

For $z > 0$

Corollary 3.2. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ the probability density function *p.d.f.* of $Z = XY$ can be expressed as:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \le 0 \\ \frac{c a^c \lambda^c \Big[\Gamma(\alpha+c) - \Gamma(\alpha+c, z/a\lambda)\Big]}{z^{c+1}\Gamma(\alpha)} & \text{if } z > 0 \end{cases} \tag{38}$$

Proof. The probability density function $f_Z(z)$ in (38) easily follows by differentiation using

1. $\frac{d}{dz}(\Gamma(\alpha, z/a\lambda)) = -\frac{z^{\alpha-1}}{(a\lambda)^\alpha} e^{-z/a\lambda}$
2. $\frac{d}{dz}(\frac{\Gamma(\alpha+c, z/a\lambda)}{z^c}) = -\frac{z^{\alpha-1} e^{-z/a\lambda}}{(a\lambda)^{\alpha+c}} - \frac{c\Gamma(\alpha+c, z/a\lambda)}{z^{c+1}}$

Corollary 3.3. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > r, \beta > 0$ the moment of order r of $Z = XY$ can be expressed as:

$$E[Z^r] = \frac{ca^c\lambda^c\Gamma(\alpha+c)\beta^{r-c}}{\Gamma(\alpha)(c-r)} - \frac{\beta^{r-c}\Gamma(\alpha+c,\beta/a\lambda)}{(c-r)} + \frac{\Gamma(r+\alpha,\beta/a\lambda)}{(c-r)(a\lambda)^{c-r}} \tag{39}$$

Proof.

$$\begin{aligned} E[Z^r] &= \int_{-\infty}^{+\infty} z^r f_Z(z)dz \\ &= \int_{\beta}^{+\infty} \frac{ca^c\lambda^c\Gamma(\alpha+c)}{\Gamma(\alpha)z^{c+1-r}}dz \\ &- \int_{\beta}^{+\infty} \frac{ca^c\lambda^c\Gamma(\alpha+c,z/a\lambda)}{\Gamma(\alpha)z^{c+1-r}}dz \end{aligned} \tag{40}$$

Let

$$I = \int_{\beta}^{+\infty} \frac{ca^c\lambda^c\Gamma(\alpha+c)}{\Gamma(\alpha)z^{c+1-r}}dz = \frac{ca^c\lambda^c\Gamma(\alpha+c)\beta^{r-c}}{\Gamma(\alpha)(c-r)} \tag{41}$$

And let

$$J = \int_{\beta}^{+\infty} \frac{ca^c\lambda^c\Gamma(\alpha+c,z/a\lambda)}{\Gamma(\alpha)z^{c+1-r}}dz \tag{42}$$

integration by part implies:

$$J = \frac{\beta^{r-c}\Gamma(\alpha+c,\beta/a\lambda)}{(c-r)} + \int_{\beta}^{\infty} \frac{z^{r+\alpha-1}e^{-z/a\lambda}}{(r-c)(a\lambda)^{\alpha+c}}dz \tag{43}$$

Using **Lemma 1** we get

$$J = \frac{\beta^{r-c}\Gamma(\alpha+c,\beta/a\lambda)}{(c-r)} - \frac{\Gamma(r+\alpha,\beta/a\lambda)}{(c-r)(a\lambda)^{c-r}} \tag{44}$$

Finally we obtain

$$E[Z^r] = \frac{ca^c\lambda^c\Gamma(\alpha+c)\beta^{r-c}}{\Gamma(\alpha)(c-r)} - \frac{\beta^{r-c}\Gamma(\alpha+c,\beta/a\lambda)}{(c-r)} + \frac{\Gamma(r+\alpha,\beta/a\lambda)}{(c-r)(a\lambda)^{c-r}} \tag{45}$$

Corollary 3.4. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > 1,\ \beta > 0.$ the Expected value of $Z = XY$ is obtained for $r = 1$ and it can be expressed as:

$$E[Z] = \frac{ca^c\lambda^c\Gamma(\alpha+c)\beta^{1-c}}{\Gamma(\alpha)(c-1)} - \frac{\beta^{1-c}\Gamma(\alpha+c,\beta/a\lambda)}{(c-1)} + \frac{\Gamma(1+\alpha,\beta/a\lambda)}{(c-1)(a\lambda)^{c-1}}. \tag{46}$$

Corollary 3.5. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > 2, \beta > 0.$ the variance of $Z = XY$ can be expressed as:

$$\begin{aligned} \sigma^2 &= \frac{ca^c\lambda^c\Gamma(\alpha+c)\beta^{2-c}}{\Gamma(\alpha)(c-2)} - \frac{\beta^{2-c}\Gamma(\alpha+c,\beta/a\lambda)}{(c-2)} + \frac{\Gamma(2+\alpha,\beta/a\lambda)}{(c-2)(a\lambda)^{c-2}} \\ &- \left[\frac{ca^c\lambda^c\Gamma(\alpha+c)\beta^{1-c}}{\Gamma(\alpha)(c-1)} - \frac{\beta^{1-c}\Gamma(\alpha+c,\beta/a\lambda)}{(c-1)} + \frac{\Gamma(1+\alpha,\beta/a\lambda)}{(c-1)(a\lambda)^{c-1}}\right]^2. \end{aligned} \tag{47}$$

Proof.
By definition, the variance of XY is:

$$\sigma^2 = E[Z^2] - E[Z]^2. \tag{48}$$

Corollary 3.6. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$. the Survival function of $Z = XY$ can be expressed as:

$$S_Z(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ \frac{\Gamma(\alpha, z/a\lambda)}{\Gamma(\alpha)} + \frac{a^c\lambda^c\Gamma(\alpha+c)}{z^c\Gamma(\alpha)} - \frac{a^c\lambda^c\Gamma(\alpha+c, z/a\lambda)}{z^c\Gamma(\alpha)} & \text{if } z > 0 \end{cases} \tag{49}$$

Proof By definition of the survival function

$$S_Z(z) = 1 - F_Z(z). \tag{50}$$

Corollary 3.7. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for the Hazard function of $Z = XY$ can be expressed as:

$$h_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{ca^c\lambda^c\left[\Gamma(\alpha+c) - \Gamma(\alpha+c, z/a\lambda)\right]}{z^{c+1}\Gamma(\alpha, z/a\lambda) + za^c\lambda^c\Gamma(\alpha+c) - za^c\lambda^c\Gamma(\alpha+c, z/a\lambda)} & \text{if } z > 0 \end{cases} \tag{51}$$

Proof By definition of the hazard function

$$h_Z(z) = \frac{f_Z(z)}{S_Z(z)}. \tag{52}$$

for $z > 0$.

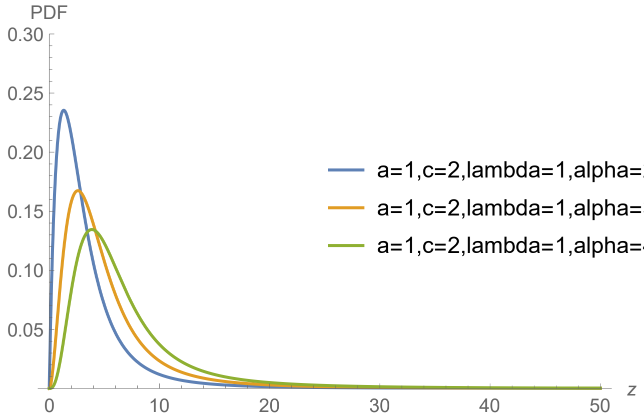

Figure 3. Plots of the pdf (38) for $a = 1, c = 2, \lambda = 1, \alpha = 2, 3, 4$.

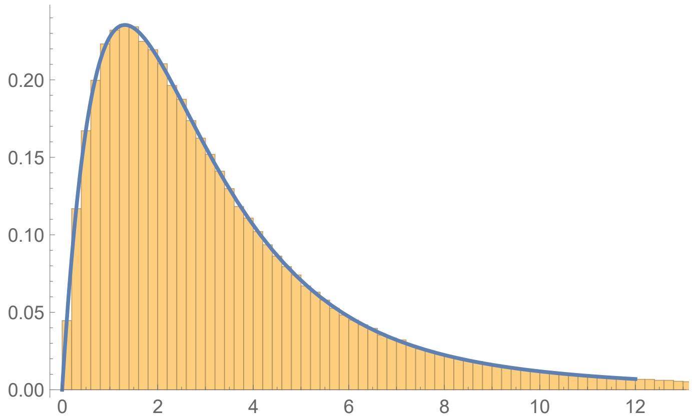

Figure 4. Monte carlo simulation for the product of pareto and erlang for $a = 1, c = 2, \lambda = 1, \alpha = 2$.

3.1. Distribution of the product of Pareto and Exponential random variables

The Erlang distribution is the distribution of a sum of α independent Exponential variables with mean $\frac{1}{b} = \lambda$ each. When $\alpha = 1$, the Erlang distribution Y simplifies to the Exponential distribution witch lead us to the following corollary.

Corollary 3.8. Suppose X and Y are two independent Pareto (1) and Erlang (2) random variables, then for $z > 0$, $\alpha = 1$. The probability density function p.d.f. of the product of Pareto and Exponential random variables $Z = XY$ can be expressed as:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{ca^c\left[\Gamma(c+1) - \Gamma(c+1, bz/a)\right]}{\lambda^c z^{c+1}} & \text{if } z > 0 \end{cases} \tag{53}$$

4. APPLICATIONS

4.1. Electric circuit

In a number of applications it is necessary to know the properties of the product of random variables: this occurs in particular when the random variables involved have dimensions of a ratio like fuel consumption per mile, cost of a structure per 1 lb. of payload, amplification ratio, tolerances expressed in percentages of the desired value, etc. Thus, for instance, if the number of accidents in a period can be regarded as a random variable and if the same applies to the number of days spent in hospital by an accident victim and to the total cost per one day-patient then the total cost is a product of these three random variables. Another application occurs, for instance, in the case when amplifiers are connected in series. If x, is the random variable describing the amplification of the *ith* amplifier then the total amplification $x = x_1x_2...x_n$ is also a random variable and it is important to know the distribution of this product.

Example, suppose an electric circuit with two amplifiers in series, X_1 is a random variable follows Pareto distribution with parameter $a = 1$, $c = 3$, and X_2 is a random variable follows Erlang distribution with parameter $\lambda = 1$,, $\alpha = 2$. then the total amplification gain is $Z = X_1.X_2$ and by using our result their pdf is

$$f_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{3(\Gamma(5)-\Gamma(5,z))}{\Gamma(2)z^4} & \text{if } z > 0 \end{cases} \tag{54}$$

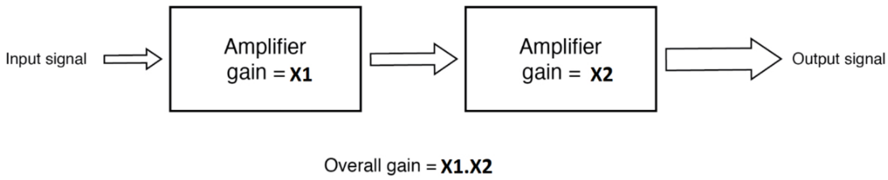

Figure 5. An electric circuit with two amplifiers in series

4.2. Portfolio of risks

In a portfolio of risks, let X denote a random variable which gives the probability that the event that claim has occurred and let Y denote the claim amount. Then the individual risk will be the product of the two random variables $Z = XY$. Assume that X follows Pareto distribution with parameters $a = 1$, $c = 3$ [**?**], and Y is a random variable follows Erlang distribution with parameters $\lambda = 1$, $\alpha = 2$ [**?**]. Using our result, the individual risk $Z = XY$ can be expressed as:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{3(\Gamma(5)-\Gamma(5,z))}{\Gamma(2)z^4} & \text{if } z > 0 \end{cases} \tag{55}$$

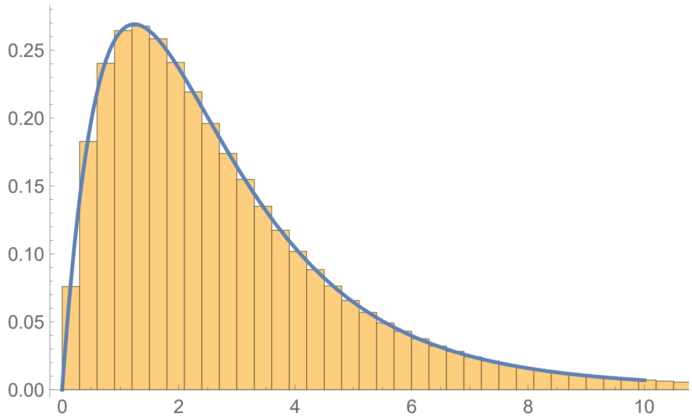

Figure 6. Monte Carlo simulation of the product XY (55), for $a = 1$, $c = 3$, $\lambda = 1$, $\alpha = 2$.

The Monte Carlo simulation code:

```
data1 = RandomVariate[ParetoDistribution[1, 3], 1000000];
data2 = RandomVariate[ErlangDistribution[2, 1], 1000000];
data3 = (data1) * (data2);
Show[
 Histogram[data3, {0.3}, "ProbabilityDensity"],
 Plot[(3 (Gamma[5] - Gamma[5, x])) / (Gamma[2] x^(4)), {x, 0, 10},
  PlotRange → {0, 1}, PlotStyle → Thick]]
```

4.3. Measurement or Radiation by Electronic Counters

Proportional, Geiger, and scintillation counters are often used to ' detect X and γ radiation, as well as other charged particles such as electrons and α particles. Design of these counters and their associated circuits depends to some extent on what is to be detected. A device common to all counters is a scaler. This electronic device counts pulses produced by the counter. Once the number of pulses over a measured period of time is known, the average counting rate is obtained by simple division. If the rate of pulses is too high for a mechanical device, it is necessary to scale down the pulses by a known factor before feeding them to a mechanical counter. There are two kinds of scalers: the binary scaler in which the scaler factor is some power of 2, and the decade scaler in which the scaling factor is some power of 10.

A typical binary scaler has several scaling factors ranging from 2^0 to 2^{14}. The scaling circuit is made up of a number of identical "stages" connected in series, the number of stages being equal to n, where 2^n is the desired scaling factor, Each stage is composed of a number of vacuum tubes, capacitors, and resistors, connected so that only one pulse of current is transmitted for every two pulses received, Since the output of one stage is connected to the input of another, this division by two is repeated as many times as there are stages. The output of the last stage is connected to a mechanical counter that will register one count for every pulse transmitted to it by the last stage. Thus, if N pulses from a counter are passed through a circuit of n stages, only $\frac{N}{2^n}$ will register on the mechanical counter. Because arrival of X-ray quanta in the counter is random in time, the accuracy of a counting rate measurement is governed by the laws of probability. Two counts of the same X-ray beam for identical periods of time will not be precisely the same because of the random spacing between pulses, even though the counter and scaler are functioning perfectly. Clearly, the accuracy of a rate measurement of this kind improves as the time of counting is prolonged. It is therefore; important to know how long to count in order to attain a specified degree of accuracy. This problem is complicated when additional background causes contamination in the counting process. This unavoidable background is due to cosmic rays and may be augmented, particularly in some laboratories, by nearby radioactive materials. Suppose we want to estimate the diffraction background in the presence of a fairly large unavoidable background. Let N be the number of pulses ! counted in a given time from a radiation source; Let N_b be the number counted in the same time with radiation source removed. The N_b counts are due to unavoidable background and $(N - N_b)$ to the diffract able background being measured. The relative probable error in $(N - N_b)$ is

$$E_{N-N_b} = \frac{67\sqrt{N + N_b}}{N - N_b}$$

percent.

since N and N_b are random variables, the desirability of obtaining the density function of the above quotient from of random variable is apparent.
For instance, if $\sqrt{N + N_b}$ is a random variable follows Pareto distribution with parameter $c = 2$ and $a = 1$, and $N - N_b$ is a random variable follows Erlang distribution with parameter $\lambda = 1, \alpha = 3$, then by using our result The relative probable error in $(N - N_b)$ is

$$E_{N-N_b} = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{134\left[\Gamma(1,1/z)\right]}{\Gamma(3)z^3} & \text{if } z > 0 \end{cases} \tag{56}$$

5. CONCLUSION

This paper has derived The analytical expressions of the PDF, CDF, the rth moment function, the variance, the survival function, and the hazard function, for the distributions of XY and X/Y when X and Y are Pareto and Erlang random variables distributed independently of each other, we illustrate our results in some graphics of the pdf for the distributions of product and ratio, finally we have discussed some examples of real-life applications for the distribution of the product and ratio, and we assured our results using Monte Carlo simulation.

Funding: This research received no external funding.
Conflict of interest: The authors declare no conflicts of interest.

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