A New 4D Hidden Hyperchaotic System with Higher Largest Lyapunov exponent and its Synchronization

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ABSTRACT: Recently, the introduction of high-dimensional systems with a larger Lyapunov exponent is very difficult and more complex. The paper introduces a new four-dimensional hyperchaotic system with a larger Lyapunov exponent compared with 20 works available in literature. The proposed system will be derived from the Lorenz-like system via a state feedback control approach and consist of eight terms only. This system lacks equilibrium points but can generate hidden attractors. Two positive Lyapunov exponents (LE) indicating hyperchaotic behavior have been identified. The mathematical properties of this dissipative hyperchaotic system are both theoretically and numerically presented, encompassing Lyapunov exponents, Lyapunov dimension (Kaplan-Yorke dimension), Multistability, and Hybrid projective synchronization (HPS). Various dynamic behavior are observed such as hyperchaotic, chaotic, chaotic 2-tours, and periodic behaviors. The paper provides proof of the main results through theoretical analysis and numerical simulations conducted in MATLAB V2021.

Keywords: Hidden attractors, Multistability, dissipative system, hybrid projective synchronization (HPS).

1. INTRODUCTION

In recent decades, there has been growing fascination with chaotic and hyperchaotic systems featuring higher-dimensional attractors. These novel chaotic systems have rapidly found applications in various scientific domains such as physics, including areas like laser dynamics[1], electrical circuits[2,3], and synchronization. Additionally, chaos theory has been utilized in both scientific and technical fields, spanning robotics[4], neural networks [5], chemical processes[6], fuzzy logic [7], encryption [8,9], and ecology [10]. This widespread application underscores the substantial interest in the study of hyperchaotic dynamics. The hidden attractor has recently become a source of inspiration for research in nonlinear science. This is due to its crucial significance in both theoretical aspects and practical engineering applications, as evidenced by a range of studies [11, 12, 13, 14].

Kuznetsov and colleagues initially proposed the concept of hidden attractors in 2010, but it gained significant recognition in 2011 through the work of Leonov and others involving Chua’s circuit[15]. Hidden attractors in dynamical systems are categorized into five types without equilibria [16], Curve equilibria[17], Curves plane[18], line of equilibria [19] and stable equilibria points [20] as illustrated in Fig. 1.

![Figure 1. Classification of hidden attractors](image-url)

Table 1 lists a compilation of prior research on high-dimensional (4D) dissipative chaotic/hyperchaotic systems, with their number of positive Lyapunov exponents, the number of terms, the maximal Lyapunov exponent (MLE), and the types of attractors. A comparative analysis between the proposed system and other different (14) 4D systems from the literature reveals that the proposed system that has fewer terms with the fulfillment of the condition \((n−2)+ve\ LEs\). Additionally, it boasts a higher largest Lyapunov exponent (MLE=3.0743). Therefore, the system is distinguished by its eight terms with hyperchaotic attractors, showcasing the maximum Lyapunov exponent (MLE).

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The main contributions of this research can be summarized in the following points:

- The proposed system has larger Lyapunov exponent compared with 20 works available.
- This system without equilibria points, so it's belong to hidden attractors.
- The system is rich in dynamic features such as hyperchaotic, chaotic, chaotic 2-tours, and periodic behaviors.
- Hybrid projective synchronization to the system is implemented.

2. THE PROPOSED SYSTEM

In 2019, Cang et al. [39] introduced a six-term 3D Lorenz-like system was developed from the 3D renowned Lorenz system [40] which described as:

\[
\begin{align*}
    \dot{x}_1 &= bx_2 - ax_1 \\
    \dot{x}_2 &= -x_1x_3 - cx_2 \\
    \dot{x}_3 &= x_1x_2 - k
\end{align*}
\]

where \(a, b\) and \(c\) are positive constants, \(x_1, x_2, x_3\) are variable states. This system exhibits chaotic behavior with one positive exponent \(LE_1=(0.103, 0, -0.653)\), under the parameters \(a = 10, b = 10, c = 100, k = 11.2\), with initial conditions \(ICs\) = \((-0.91, 1.94, 0.86)\) and the corresponding Lyapunov dimension is \(D_L = 2.1577\). The equilibria points \(E_1(\sqrt{\frac{bk}{a}}, \sqrt{\frac{ak}{b}}, -\frac{ak}{b})\) and \(E_2(-\sqrt{\frac{bk}{a}}, -\sqrt{\frac{ak}{b}}, \frac{ak}{b})\), indicate that system (1) is of the self-excited and hidden attractors type and is of a dissipative nature.

Based on feedback control strategy, a new 4D hyperchaotic system is presented which depicted as:
\[
\begin{align*}
\dot{x}_1 &= bx_2 - ax_1 \\
\dot{x}_2 &= -x_1x_3 - cx_2 - dx_4 \\
\dot{x}_3 &= x_1x_2 - k \\
\dot{x}_4 &= x_1
\end{align*}
\] (2)

where \(d, k \neq 0\) are control parameters.

To find the equilibrium points for system (2), assume \((\dot{x}_1 = \cdots = \dot{x}_4 = 0)\), so we get from the fourth equation: \(x_1 = 0\). Substituting this value into the third equation, we find that \(k = 0\). This contradicts the hypothesis, so this system is classified as a hidden system.

The two parameters \(a\) and \(c\) are important in classifying the nature of the system (2) as either conservative or dissipative, depending on the Trace of the matrix, as follows:

\[
tr = \sum_{i=1}^{4} \frac{\partial \dot{x}_i}{\partial x_i} = -a - c
\]

- If \(a = -c\), then system (2) is conservative.
- If \(a > -c\), then system (2) is dissipative.
- If \(a < -c\), then system (2) is unbounded.

2.1. Exponents and dimension Lyapunov

Different dynamical behavior are observed such as chaotic 2-torus, periodic, chaotic, and hyperchaotic which identified via the sign of the LE as illustration in Table 2 and Fig. 2. Based upon the Lyapunov exponents, the dynamical behaviors of the system (2) may be divided into the following categories, as shown in Table 3.

Table 2. Lyapunov Exponents with parameters \(a = b = 10, c = 0.3, k = 100\), and different \(d\), with IC (4).

<table>
<thead>
<tr>
<th>Fig. 4a</th>
<th>Parameters</th>
<th>Lyapunov Exponents ((LE_1, LE_2, LE_3, LE_4))</th>
<th>Sign of (LEs)</th>
<th>Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(d = 0.3)</td>
<td>((2.0133, 0.0013, -0.0004, -12.3087))</td>
<td>((+, +, 0, -))</td>
<td>Hyperchaotic</td>
</tr>
<tr>
<td>Fig. 4b</td>
<td>(d = 0.0063)</td>
<td>((2.0330, -0.0002, -0.0016, -12.3257))</td>
<td>((+, 0, 0, -))</td>
<td>Chaotic</td>
</tr>
<tr>
<td>Fig. 4c</td>
<td>(d = 0.0683)</td>
<td>((2.0190, -0.0008, -0.0001, -12.3128))</td>
<td>((+, 0, 0, -))</td>
<td>Chaotic 2-torus</td>
</tr>
<tr>
<td>Fig. 4d</td>
<td>(d = 100.5)</td>
<td>((0.0003, -0.0014, -0.0392, -10.2599))</td>
<td>((0, 0, 0, -))</td>
<td>Periodic</td>
</tr>
</tbody>
</table>

Table 3. Behaviors of a 4D dynamical system have the different LE sign.

<table>
<thead>
<tr>
<th>(LE_1)</th>
<th>(LE_2)</th>
<th>(LE_3)</th>
<th>(LE_4)</th>
<th>Type of Attractors</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>0</td>
<td>–</td>
<td>Hyperchaotic</td>
</tr>
<tr>
<td>+</td>
<td>0</td>
<td>0</td>
<td>–</td>
<td>Chaotic (2-tours)</td>
</tr>
<tr>
<td>+</td>
<td>0</td>
<td>–</td>
<td>–</td>
<td>Chaotic</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>–</td>
<td>Quasi-periodic (3-torus)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>–</td>
<td>–</td>
<td>Quasi-periodic (2-tours)</td>
</tr>
<tr>
<td>0</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>Periodic (limit cycles)</td>
</tr>
<tr>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>Solution approach fixed points</td>
</tr>
</tbody>
</table>
Figure 2. Attractors system (2) under various parameters (a) Hyperchaotic, (b) Chaotic, (c) Chaotic 2-tours, (d) Periodic

By using the Wolf algorithm [41] and under the parameters (3) and initial conditions (4), the new system (2) has two positive $LE$, as depicted in Fig. 3, therefore referred to as having "hyperchaotic attractors" as seen in Fig. 4.

\[
\begin{align*}
    a &= 10 \\
    b &= 100 \\
    c &= 10 \\
    d &= 0.27 \\
    k &= 100 \\
\end{align*}
\]

\[X(0) = (0.91, 1.94, 0.86, 0.3)\]

\[
\begin{align*}
    LE_1 &= 3.0743 \\
    LE_2 &= 0.0068 \\
    LE_3 &= -0.0005, \quad \Sigma_{i=1}^{4} LE_i = -19.9573 \\
    LE_4 &= -23.0379
\end{align*}
\]

It is clear that $\Sigma_{i=1}^{4} LE_i = -19.9573 \cong tr = -20$, this indicates the validity of the theoretical results with the numerical solutions of the Wolf algorithm. In addition the system (2) has nature dissipative. Determine the complexity of a new system's attractor by determining the Lyapunov dimension ($D_L$), which is described as [42]:

\[
D_L = J + \frac{1}{|LE_{i+1}|} \sum_{i=1}^{t} LE_i
\]

\[
D_L = 3 + \frac{3.0743 + 0.0068 - 0.0005}{23.0379} = 3.1337
\]
2.2. Multistability

Multistability, a feature of complex systems such as systems with feedback loops or non-linear systems, is a fundamental idea in engineering, physics, economics, and biology. It is critical for understanding phenomena including phase transitions, decision-making processes, pattern formation, and cell differentiation. Understanding and studying multistability within dynamical systems is critical because it provides deep insights into the system’s behavior and assists in the creation of control techniques aimed at steering the system towards certain attractors or states. Non-linear dynamical systems’ multistability indicates that they can have several solutions within the same parameters. Table 4 shows two attractors with the same parameters and under different initial conditions, whereas Fig. 5 exhibits the corresponding attractors (solutions).

Table 4. Multistability with different Parameters and Initial Conditions

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Figure</th>
<th>Initial Conditions</th>
<th>Color</th>
</tr>
</thead>
<tbody>
<tr>
<td>a = c = 10</td>
<td>Fig.5a</td>
<td>(3, 1.9, 0.86, 0.3)</td>
<td>Red</td>
</tr>
<tr>
<td>b = 100</td>
<td></td>
<td>(−1, −0.4, 0.4, 0.5)</td>
<td>Blue</td>
</tr>
<tr>
<td>d = 0.27</td>
<td></td>
<td>(1.0, 0.4, 0.4, 0.5)</td>
<td>Green</td>
</tr>
<tr>
<td>k = 100</td>
<td></td>
<td>(0.01, 0.5, 0.9, 0.1)</td>
<td>Red</td>
</tr>
<tr>
<td></td>
<td>Fig.5b</td>
<td>(−0.01, −0.5, −0.8, −1)</td>
<td>Blue</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(−40, 0.7, 0.7, 0.7)</td>
<td>Red</td>
</tr>
<tr>
<td></td>
<td>Fig.5c</td>
<td>(40, 0.7, 0.7, 0.7)</td>
<td>Magenta</td>
</tr>
<tr>
<td>a = c = 10</td>
<td>Fig.5d</td>
<td>(−10, −7, 7, 7)</td>
<td>Blue</td>
</tr>
<tr>
<td>b = 1</td>
<td></td>
<td>(10, 7, 7, 7)</td>
<td>Magenta</td>
</tr>
</tbody>
</table>
2.3. Compare results

By comparing between the new system and the original system through Table 5, can be seen that the new system (2) is more effective than the system (1) in terms of both the Lyapunov exponents, behavior of system, and the convergence between Lyapunov's exponential sum ($\sum_{i=1}^{n} LE_i$) and the Trace of the matrix ($tr$).

<table>
<thead>
<tr>
<th>Details</th>
<th>3D System (1)</th>
<th>New 4D system (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. $LE$</td>
<td>0.103</td>
<td>3.0743</td>
</tr>
<tr>
<td>Lyapunov dimension</td>
<td>2.1577</td>
<td>3.1337</td>
</tr>
<tr>
<td>$n - 2$</td>
<td>(n - 2)</td>
<td></td>
</tr>
<tr>
<td>No. $+$ve $LE$</td>
<td>--------</td>
<td></td>
</tr>
<tr>
<td>Behavior of System</td>
<td>Chaotic (+,0,-)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Hyperchaotic (+,+,0,-)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Chaotic(+,0,-)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Chaotic 2-torus (+,0,0,-)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Periodic (0,-,0,-)</td>
<td></td>
</tr>
</tbody>
</table>

3. HYBRID PROJECTIVE SYNCHRONIZATION (HPS)

Hybrid synchronization (HS) is synchronization that combines complete synchronization (CS) with anti-synchronization (AS) and is a special case of hybrid projective synchronization (HPS). In this section, the problem of hybrid projective synchronization (HPS) are implemented on the proposed system. The HPS consists of two nonlinear dynamical systems: the drive system and the response system. The response system has control over the drive system. The mathematical formulations for the drive and response systems are given as equations (5)(from system (2)) and (6), respectively [43].
A Lyapunov function is built as:

\[
\begin{align*}
\dot{X}_1 &= \begin{bmatrix} a & b & 0 & 0 \\ 0 & -c & 0 & -d \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}X_1 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -k \\ 0 & 0 & 0 & 0 \end{bmatrix}X_1X_3 + \begin{bmatrix} x_1 \ x_2 \ x_3 \ 1 \end{bmatrix} \\
\dot{Y}_1 &= \begin{bmatrix} a & b & 0 & 0 \\ 0 & -c & 0 & -d \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}Y_1 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -k \\ 0 & 0 & 0 & 0 \end{bmatrix}Y_1Y_3 + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}
\end{align*}
\]

Furthermore, the error dynamical system is described as:

\[
e_i = y_i - P x_i
\]

\(P\) is n-order diagonal matrix, when \(i = 1, 2, 3, 4\), \(P = diag(p_1, p_2, p_3, p_4)\), the concept of a scaling matrix "\(P\)" its associated scaling factors "\(p_1, ..., p_n\)". The objective is to design a controller "\(U\)" that guides the response system to asymptotically match the drive system. This alignment leads to synchronization, where the error between the two systems converges to zero over time.

\[
\lim_{t \to \infty} \|e_i(t)\| = \lim_{t \to \infty} \|y_i - P x_i\| = 0
\]

The significance of the scaling matrix \(P\) lies in its role in determining the synchronization phenomenon, like if \(P\) is a matrix that is constant and:

- If \(p_1 \neq p_2 \neq \cdots \neq p_n\), then is called a hybrid projective synchronization (HPS).
- If \(p_1 = p_2 = \cdots = p_n\), and \((\forall p_i = -1)\) and \((\forall p_i = 1)\), then is called a complete synchronization (CS) and Anti-synchronization (AS), respectively.
- If \(\forall p_i = \pm 1\), then is called a Hybrid Synchronization (HS).

### 3.1. Controllers for scaling factor \(p_1 = 4, p_2 = 2, p_3 = 3, p_4 = 1\)

If matrix \(P\) is selected as \(P = diag(4, 2, 3, 1)\), i.e.:

\[
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
\]

Adding system (5) to system (6) yields an error system (9):

\[
\begin{align*}
\dot{e}_1 &= be_2 - ae_1 - 2bx_2 + u_1 \\
\dot{e}_2 &= -ce_2 - d(e_4 - x_4) - e_3(e_1 + 4x_1) - x_3(3e_1 + 10x_1) + u_2 \\
\dot{e}_3 &= e_2(e_1 + 4x_1) + x_2(2e_1 + 5x_1) + 2k + u_3 \\
\dot{e}_4 &= e_1 + 3x_1 + u_4
\end{align*}
\]

**Theorem 1:** A designed non-linear controller as

\[
\begin{align*}
u_1 &= -e_4 + x_2(2b - 2e_3) + e_2(3x_4 - e_3) \\
u_2 &= -be_1 - dx_4 + 10x_1x_3 \\
u_3 &= -e_3 + e_1e_2 - 2k - 5x_1x_2 \\
u_4 &= -e_4 + de_2 - 3x_1
\end{align*}
\]

Then takes place Hybrid Synchronization between systems (5) and (6).

**Proof:** Inserting (11) in (10) we have:

\[
\begin{align*}
\dot{e}_1 &= -ae_1 + e_2(b + 3x_3 - e_2) - 2e_3x_2 - e_4 \\
\dot{e}_2 &= -ce_2 - de_4 - e_3(e_1 + 4x_1) - e_1(3x_3 + b) \\
\dot{e}_3 &= -e_3 + e_2(2e_1 + 4x_1) + 2e_1x_2 \\
\dot{e}_4 &= -e_4 + e_1 + de_2
\end{align*}
\]

A Lyapunov function is built as: \(V(e_i) = e_i^TSe_i\)
\( V(e_i) = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \) (13)

The time derivative of a preceding function \( V(e_i) \) is given by:

\[
\dot{V}(e_i) = e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_4 \dot{e}_4
\]

(14)

\[
\dot{V}(e_i) = -ae_i^2 - ce_i^2 - e_i^3 - e_i^4
\]

(15)

As a result, \( \dot{V}(e_i) \) is negative definite (where \( a = c = 10 \)), so the hybrid synchronization process has been achieved theoretically with control system (11).

3.2. Controllers for scaling factor \( p_1 = 1, p_2 = -1, p_3 = 1, p_4 = -1 \)

If matrix \( P \) is selected as \( P = \text{diag}(1, -1, 1, -1) \), i.e.:

\[
e = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
\]

(16)

Adding system (5) to system (6) yields an error system (16):

\[
\begin{align*}
\dot{e}_1 &= -ae_1 + be_2 - 2bx_2 + u_1 \\
\dot{e}_2 &= -ce_2 - de_4 - e_3(e_1 + x_1) - x_3(e_1 + 2x_1) + u_2 \\
\dot{e}_3 &= e_2(e_1 + x_1) - x_2(e_1 + 2x_1) + u_3 \\
\dot{e}_4 &= e_1 + u_4
\end{align*}
\]

Theorem 2: A designed non-linear controller as

\[
\begin{align*}
u_1 &= -e_4 + x_2(2b + e_3) + e_2x_3 \\
u_2 &= -be_3 + 2x_1x_3 - e_1e_3 \\
u_3 &= -e_3 + e_4e_2 + 2x_1x_2 \\
u_4 &= -e_4 + de_2
\end{align*}
\]

Then takes place Hybrid Synchronization between systems (5) and (6).

Proof: Inserting (18) in (17) we have:

\[
\begin{align*}
\dot{e}_1 &= -ae_1 + be_2 - e_4 + x_2e_3 + x_3e_2 \\
\dot{e}_2 &= -ce_2 - de_4 - e_3(2e_1 + x_1) - x_3e_1 - be_1 \\
\dot{e}_3 &= e_2(2e_1 + x_1) - x_2e_1 - e_3 \\
\dot{e}_4 &= -e_4 + e_1 + de_2
\end{align*}
\]

(19)

The time derivative (14) of a preceding function \( V(e_i) \) is given by:

\[
\dot{V}(e_i) = -ae_i^2 - ce_i^2 - e_i^3 - e_i^4
\]

(20)

As a result, \( \dot{V}(e_i) \) is negative definite, so the hybrid synchronization process has been achieved theoretically with control system (18).

3.3. Numerical simulation

By applying the fourth-order Runge-Kutta scheme and for using a time step of 0.01 and the initial condition of both the drive system and the response system are following \((0.3,0.2,0.1,0.2)\) and \((-0.1,0.3,-0.4,0.1)\) respectively, the differential equations of the controlled error dynamical system (10) for PHS and the controlled error dynamical system (17) for HS is adopted determined throughout simulation using MATLAB R2021.
• For the scaling factor $p_1 = 4, p_2 = 2, p_3 = 3, p_4 = 1$. The HPS for systems (5) and (6) through control (11) is shown in Figs. 5 and 6.

• For the scaling factor $p_1 = 1, p_2 = -1, p_3 = 1, p_4 = -1$. The HPS for systems (5) and (6) through control (18) is shown in Figs. 7 and 8.

Figure 2. Hybrid Synchronization with and without error controllers (11).

Figure 3. The HPS for the state variables with control (11) at scaling factors $p_1 = 4, p_2 = 2, p_3 = 3, p_4 = 1$.
Figure 4. Hybrid Synchronization with and without error controllers (18).

Figure 5. The HS for the state variables with control (18) at scaling factors $p_1 = 1, p_2 = -1, p_3 = 1, p_4 = -1$

4. CONCLUSION

The work's main goal was to explore hidden attractors and analyze complex nonlinear dissipative hyperchaotic systems. This was achieved by employing a state feedback method involving an eight-term system. Notably, the system features a maximal Lyapunov exponent (MLE) of 3.0743, surpassing the Lyapunov exponent of the original system. Various characteristics were explored, including the Lyapunov dimension, multistability, and hybrid projective synchronization (HPS), which is valuable across science, engineering, and technology. The system’s potential applications span science, technology, neural networks, and robotics, with MATLAB V2021 simulations used to present the key findings.
REFERENCES


