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A New 4D Hidden Hyperchaotic System with Higher Largest Lyapunov exponent and its Synchronization

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ABSTRACT: Recently, the introduction of high-dimensional systems with a larger Lyapunov exponent is very difficult and more complex. The paper introduces a new four-dimensional hyperchaotic system with a larger Lyapunov exponent compared with 20 works available in literature. Th proposed system will be derived from the Lorenz-like system via a state feedback control approach and consist of eight terms only. This system lacks equilibrium points but can generate hidden attractors. Two positive Lyapunov exponents (LE) indicating hyperchaotic behavior have been identified. The mathematical properties of this dissipative hyperchaotic system are both theoretically and numerically presented, encompassing Lyapunov exponents, Lyapunov dimension (Kaplan-Yorke dimension), Multistability, and Hybrid projective synchronization (HPS). Various dynamic behavior are observed such as hyperchaotic, chaotic, chaotic 2-tours, and periodic behaviors. The paper provides proof of the main results through theoretical analysis and numerical simulations conducted in MATLAB V2021.

Keywords: Hidden attractors, Multistability, dissipative system, hybrid projective synchronization (HPS).

1. INTRODUCTION

In recent decades, there has been growing fascination with chaotic and hyperchaotic systems featuring higher-dimensional attractors. These novel chaotic systems have rapidly found applications in various scientific domains such as physics, including areas like laser dynamics[1], electrical circuits[2],[3], and synchronization. Additionally, chaos theory has been utilized in both scientific and technical fields, spanning robotics[4], neural networks [5], chemical processes[6], fuzzy logic [7], encryption [8],[9], and ecology [10]. This widespread application underscores the substantial interest in the study of hyperchaotic dynamics. The hidden attractor has recently become a source of inspiration for research in nonlinear science. This is due to its crucial significance in both theoretical aspects and practical engineering applications, as evidenced by a range of studies [11], [12], [13], [14].

Kuznetsov and colleagues initially proposed the concept of hidden attractors in 2010, but it gained significant recognition in 2011 through the work of Leonov and others involving Chua's circuit[15]. Hidden attractors in dynamical systems are categorized into five types without equilibria [16], Curve equilibria[17], Curves plane [18], line of equilibria [19] and stable equilibria points [20] as illustrated in Fig. 1.

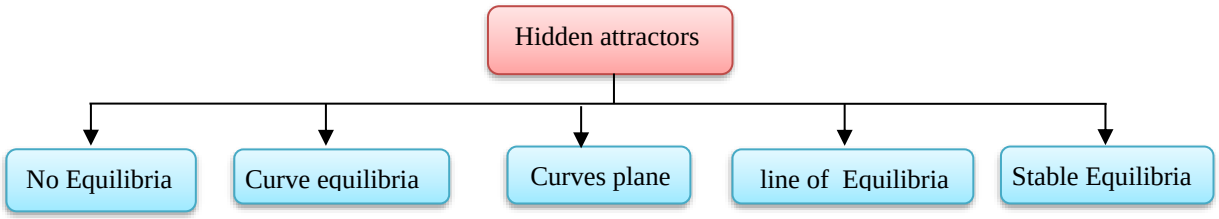

Figure 1. Classification of hidden attractors

Table 1 lists a compilation of prior research on high-dimensional (4D) dissipative chaotic/hyperchaotic systems, with their number of positive Lyapunov exponents, the number of terms, the maximal Lyapunov exponent (MLE), and the types of attractors. A comparative analysis between the proposed system and other different (14) 4D systems from the literature reveals that the proposed system that has fewer terms with the fulfillment of the condition $(n-2)+ve\ LEs$. Additionally, it boasts a higher largest Lyapunov exponent (MLE=3.0743). Therefore, the system is distinguished by its eight terms with hyperchaotic attractors, showcasing the maximum Lyapunov exponent (MLE).

Table 1. Categorization of most 4D dynamical systems.

No.	Behavior of System	$+ve$ LEs	No. Terms	maximal Lyapunov exponent MLE	attractor types	Refs.
1	Periodic	$n-4$	10	-0.0006	Self-excited	2020, [20]
2	Hyperchaotic	$n-2$	14	0.0179	Self-excited	2019, [21]
3	Hyperchaotic	$n-2$	12	0.5516	Self-excited	2021, [22]
4	Hyperchaotic	$n-2$	11	0.03443	Self-excited	2012, [23]
5	Hyperchaotic	$n-2$	10	0.8367	Hidden	2021, [24]
6	Chaotic	$n-3$	10	0.0220	Hidden	2020, [20]
7	Hyperchaotic	$n-2$	9	0.8700	Hidden	2012, [25]
8	Hyperchaotic	$n-2$	10	0.2661	Self-excited	2016, [26]
9	Chaotic	$n-3$	8	0.0360	Hidden	2017, [27]
10	Chaotic	$n-3$	7	0.1888	Hidden	2020, [28]
11	Hyperchaotic	$n-2$	9	0.7796	Hidden	2022, [29]
12	Chaotic	$n-3$	8	0.1658	Hidden	2021, [30]
13	Hyperchaotic	$n-2$	9	0.2018	Multiple	2022, [31]
14	Chaotic	$n-3$	7	1.0714	Hidden	2018, [32]
15	Chaotic	$n-3$	9	0.0555	Hidden	2017, [33]
16	Chaotic 2-torus	$n-3$	8	0.0568	Hidden	2018, [34]
17	Chaotic	$n-3$	8	0.11941	Hidden	2018, [35]
18	Chaotic	$n-3$	9	0.5888	Self-excited	2021, [36]
19	Hyperchaotic	$n-2$	11	1.0986	Hidden	2022, [37]
20	Chaotic	$n-3$	8	N/V	Self-excited	2022, [38]
21	Hyperchaotic	$\boldsymbol{n-2}$	**8**	**3.0743**	Hidden	This work

The main contributions of this research can be summarized in the following points:

- The proposed has larger Lyapunov exponent compared with 20 works available,
- This system without equilibria points, so its belong to hidden attractors.
- The system is rich in dynamic features such as hyperchaotic, chaotic, chaotic 2-tours, and periodic behaviors.
- Hybrid projective synchronization to the system is implemented.

2. THE PROPOSED SYSTEM

In 2019, Cang et al. [39] introduced a six-term 3D Lorenz-like system was developed from the 3D renowned Lorenz system [40] which described as:

$$\begin{cases} \dot{x}_1 = bx_2 - ax_1 \\ \dot{x}_2 = -x_1x_3 - cx_2 \\ \dot{x}_3 = x_1x_2 - k \end{cases} \quad (1)$$

where a, b and c are positive constants, x_1, x_2, x_3 are variable states. This system exhibits chaotic behavior with one positive exponent LE_i=$(0.103, 0, -0.653)$, under the parameters $a = 10, b = 10, c = 100, k = 11.2$, with initial conditions $(ICs) = (-0.91, 1.94, 0.86)$ and the corresponding Lyapunov dimension is $D_L = 2.1577$. The equilibria points $E_1(+\sqrt{\frac{bk}{a}}, +\sqrt{\frac{ak}{b}}, -\frac{ac}{b})$ and $E_2(-\sqrt{\frac{bk}{a}}, -\sqrt{\frac{ak}{b}}, -\frac{ac}{b})$, indicate that system (1) is of the self-excited and hidden attractors type and is of a dissipative nature.

Based on feedback control strategy, a new 4D hyperchaotic system is presented which depicted as:

$$\begin{cases} \dot{x}_1 = bx_2 - ax_1 \\ \dot{x}_2 = -x_1x_3 - cx_2 - dx_4 \\ \dot{x}_3 = x_1x_2 - k \\ \dot{x}_4 = x_1 \end{cases} \quad (2)$$

where $d, k \neq 0$ are control parameters.

To find the equilibrium points for system (2), assume ($\dot{x}_1 = \cdots = \dot{x}_4 = 0$), so we get from the fourth equation: $x_1 = 0$. Substituting this value into the third equation, we find that $k = 0$. This contradicts the hypothesis, so this system is classified as a hidden system.

The two parameters a and c are important in classifying the nature of the system (2) as either conservative or dissipative, depending on the Trace of the matrix, as follows:

$$tr = \sum_{i=1}^{4} \frac{\partial \dot{x}_i}{\partial x_i} = -a - c$$

- If $a = -c$, then system (2) is conservative.
- If $a > -c$, then system (2) is dissipative.
- If $a < -c$, then system (2) is unbounded.

2.1. Exponents and dimension Lyapunov

Different dynamical behavior are observed such as chaotic 2-torus, periodic, chaotic, and hyperchaotic which identified via the sign of the LE as illustration in Table 2 and Fig. 2. Based upon the Lyapunov exponents, the dynamical behaviors of the system (2) may be divided into the following categories, as shown in Table 3.

Table 2. Lyapunov Exponents with parameters $a = b = 10, c = 0.3, k = 100$, and different d, with IC (4).

Fig.4	Parameters	Lyapunov Exponents (LE_1, LE_2, LE_3, LE_4)	Sign of LEs	Behavior
Fig.4*a*	$d = 0.3$	$(2.0133, 0.0013, \mathbf{-0.0004}, -12.3087)$	$(+,+,0,-)$	Hyperchaotic
Fig.4*b*	$d = 0.0063$	$(2.0330, \mathbf{0.0002}, -0.0016, -12.3257)$	$(+,0,-,-)$	Chaotic
Fig.4*c*	$d = 0.0683$	$(2.0190, \mathbf{-0.0008}, \mathbf{-0.0001}, -12.3128)$	$(+,0,0,-)$	Chaotic 2-torus
Fig.4*d*	$d = 100.5$	$(\mathbf{0.0003}, -0.0014, -0.0392, -10.2599)$	$(0,-,-,-)$	Periodic

Table 3. Behaviors of a 4D dynamical system have the different LE sign.

LE_1	LE_2	LE_3	LE_4	Type of Attractors
+	+	0	−	Hyperchaotic
+	0	0	−	Chaotic (2-tours)
+	0	−	−	Chaotic
0	0	0	−	Quasi-periodic (3-torus)
0	0	−	−	Quasi-periodic (2-tours)
0	−	−	−	Periodic (limit cycles)
−	−	−	−	Solution approach fixed points

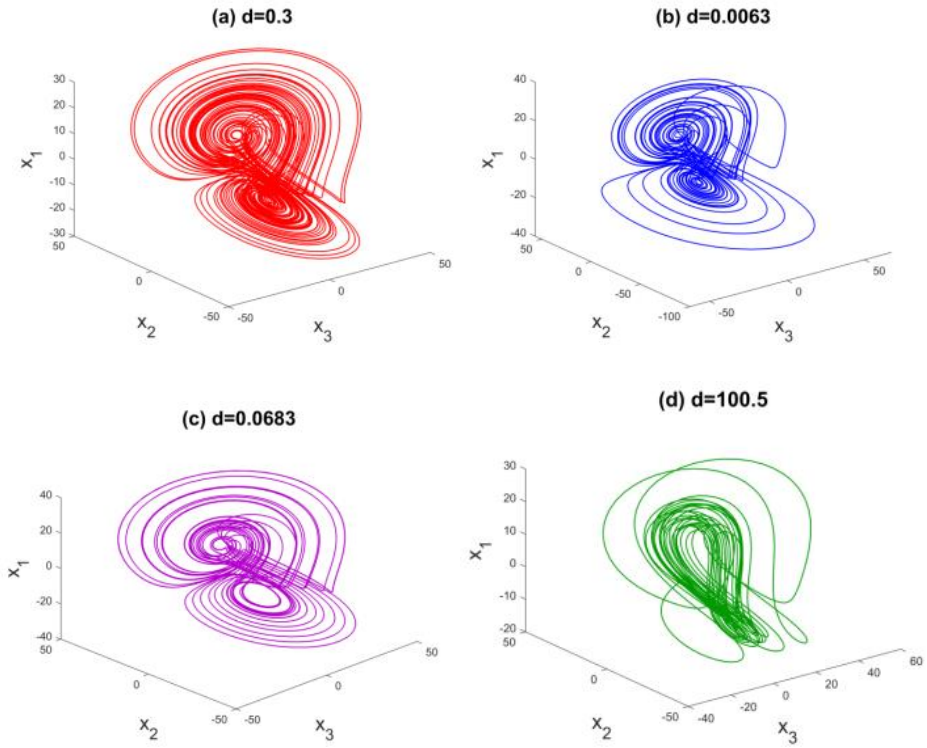

Figure 2. Attractors system (2) under various parameters (a) Hyperchaotic, (b) Chaotic, (c) Chaotic 2-tours, (d) Periodic

By using the Wolf algorithm [41] and under the parameters (3) and initial conditions (4), the new system (2) has two positive LE, as depicted in Fig. 3, therefore referred to as having "hyperchaotic attractors" as seen in Fig. 4.

$$\begin{cases} a = 10 \\ b = 100 \\ c = 10 \\ d = 0.27 \\ k = 100 \end{cases} \tag{3}$$

$$X(0) = (0.91, 1.94, 0.86, 0.3) \tag{4}$$

$$\begin{cases} LE_1 = 3.0743 \\ LE_2 = 0.0068 \\ LE_3 = -\mathbf{0.0005} \\ LE_4 = -23.0379 \end{cases}, \quad \sum_{i=1}^{4} LE_i = -19.9573$$

It is clear that $\sum_{i=1}^{4} LE_i = -19.9573 \cong tr = -20$, this indicates the validity of the theoretical results with the numerical solutions of the Wolf algorithm. In addition the system (2) has nature dissipative. Determine the complexity of a new system's attractor by determining the Lyapunov dimension (D_L), which is described as [42]:

$$D_L = J + \frac{1}{|LE_{J+1}|}\sum_{i=1}^{J} LE_i$$

$$D_L = 3 + \frac{3.0743 + 0.0068 - 0.0005}{23.0379} = 3.1337$$

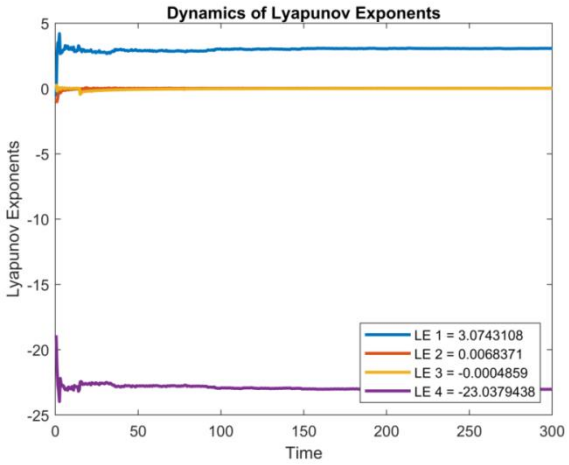

Figure 3. The system's two positive Lyapunov exponents (2) with typical parameters (3), and IC (4).

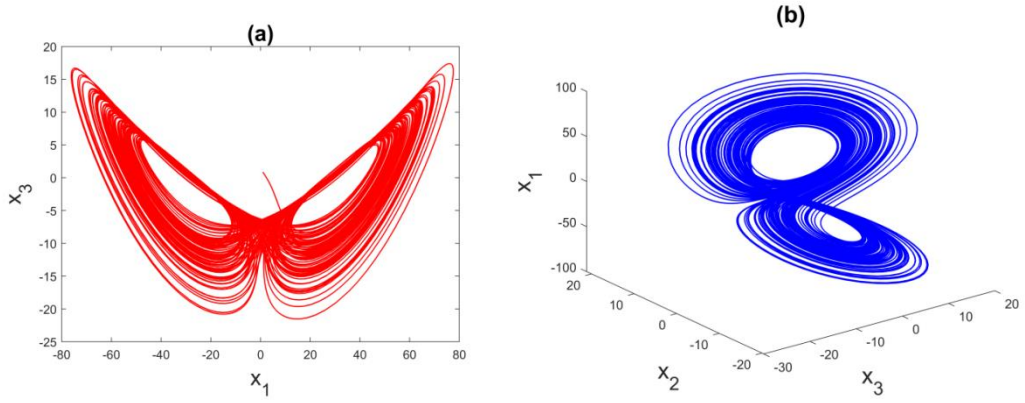

Figure 4. The phase portrait for the new system

2.2. Multistability

Multistability, a feature of complex systems such as systems with feedback loops or non-linear systems, is a fundamental idea in engineering, physics, economics, and biology. It is critical for understanding phenomena including phase transitions, decision-making processes, pattern formation, and cell differentiation. Understanding and studying multistability within dynamical systems is critical because it provides deep insights into the system's behavior and assists in the creation of control techniques aimed at steering the system towards certain attractors or states. Non-linear dynamical systems' multistability indicates that they can have several solutions within the same parameters. Table 4 shows two attractors with the same parameters and under different initial conditions, whereas Fig. 5 exhibits the corresponding attractors (solutions).

Table 4. Multistability with different Parameters and Initial Conditions

Parameters	Figure	Initial Conditions	Color
$a = c = 10$ $b = 100$ $d = 0.27$ $k = 100$	Fig.5a	$(3, 1.9, 0.86, 0.3)$	Red
		$(-1, -0.4, 0.4, 0.5)$	Blue
		$(1, 0.4, 0.4, 0.5)$	Green
	Fig.5b	$(0.01, 0.5, 0.8, 0.1)$	Red
		$(-0.01, -0.5, -0.8, -1)$	Blue
	Fig.5c	$(-40, 0.7, 0.7, 0.7)$	Red
		$(40, 0.7, 0.7, 0.7)$	Magenta
$a = c = 10$ $b = 10$ $d = 0.2$ $k = 10$	Fig.5d	$(-10, -7, 7, 7)$	Blue
		$(10, 7, 7, 7)$	Magenta

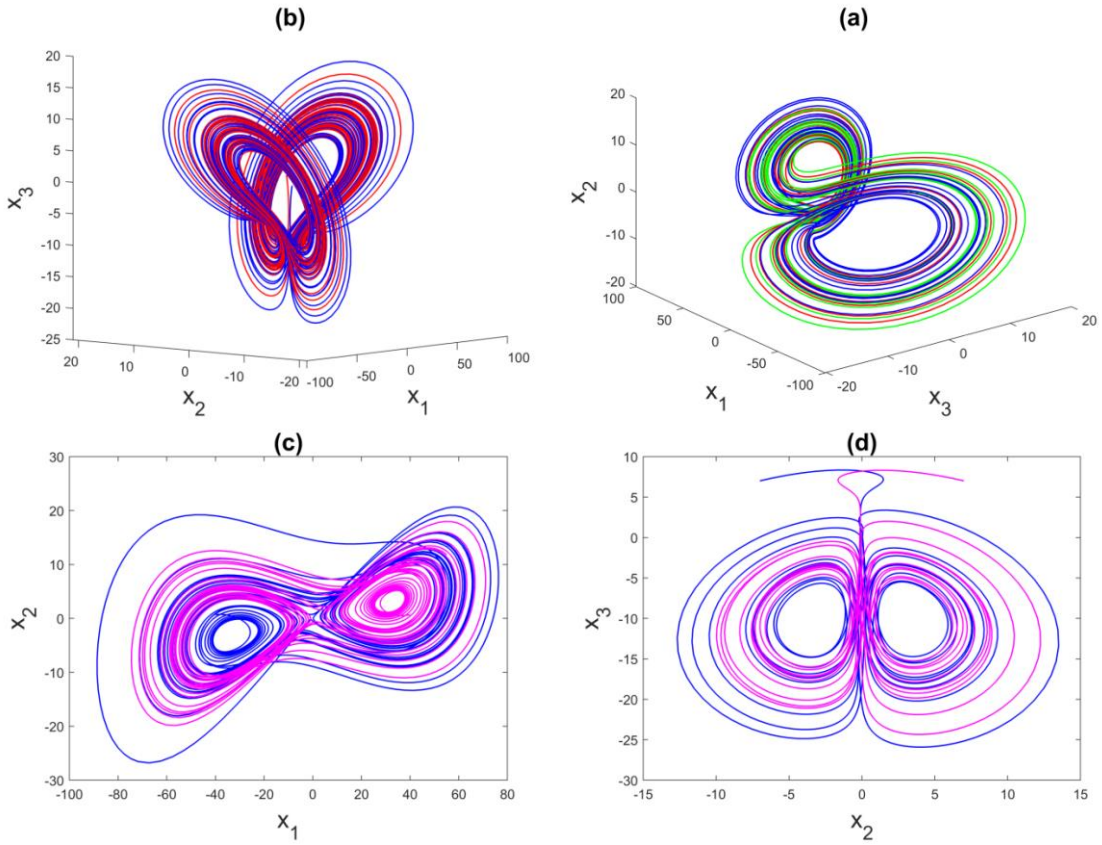

Figure 5. The coexistence attractor of a system (2) in (a): $x_3 - x_1 - x_2$ space, (b): $x_1 - x_2 - x_3$space, (c): $x_1 - x_2$ plane, (d): $x_2 - x_3$ plane.

2.3. Compare results

By comparing between the new system and the original system through Table 5, can be seen that the new system (2) is more effective than the system (1) in terms of both the Lyapunov exponents, behavior of system, and the convergence between Lyapunov's exponential sum $(\sum_{i=1}^{i} LE_i)$ and the Trace of the matrix (tr).

Table 5. Comparison results between system (1) and the proposed system.

Details	3D System (1)	New 4D system (2)
Max. LE	0.103	$\mathbf{3.0743}$
Lyapunov dimension	2.1577	3.1337
No. $+ve$ *LE*	--------	$(\boldsymbol{n-2})$
Behavior of System	Chaotic (+,0,-)	• Hyperchaotic (+,+,0,-) • Chaotic(+,0,-,-) • Chaotic 2-torus (+,0,0,-) • Periodic (0,-,-,-)

3. HYBRID PROJECTIVE SYNCHRONIZATION (HPS)

Hybrid synchronization (HS) is synchronization that combines complete synchronization (CS) with anti-synchronization (AS) and is a special case of hybrid projective synchronization (HPS). In this section, the problem of hybrid projective synchronization (HPS) are implemented on the proposed system. The HPS consists of two nonlinear dynamical systems: the drive system and the response system. The response system has control over the drive system. The mathematical formulations for the drive and response systems are given as equations (5)(from system (2)) and (6), respectively [43].

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}-a & b & 0 & 0\\ 0 & -c & 0 & -d\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & -k\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1x_3\\ x_1x_2\\ 1\\ 0\end{bmatrix} \quad (5)$$

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\end{bmatrix}=\begin{bmatrix}-a & b & 0 & 0\\ 0 & -c & 0 & -d\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & -k\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}y_1y_3\\ y_1y_2\\ 1\\ 0\end{bmatrix}+\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix} \quad (6)$$

Furthermore, the error dynamical system is described as:

$$e_i = y_i - P\,x_i \quad (7)$$

P is n-order diagonal matrix, when $i = 1, 2, 3, 4$, $P = diag(p_1, p_2, p_3, p_4)$, the concept of a scaling matrix "P" its associated scaling factors "$p_1, \dots, p_n$". The objective is to design a controller "U" that guides the response system to asymptotically match the drive system. This alignment leads to synchronization, where the error between the two systems converges to zero over time.

$$\lim_{t\to\infty}\|e_i(t)\| = \lim_{t\to\infty}\|y_i - P\,x_i\| = 0 \quad (8)$$

The significance of the scaling matrix P lies in its role in determining the synchronization phenomenon, like if P is a matrix that is constant and:

- If $p_1 \neq p_2 \neq \cdots \neq p_n$, then is called a hybrid projective synchronization (HPS),
- If $p_1 = p_2 = \cdots = p_n$, and ($\forall pi = -1$) and ($\forall pi = 1$), then is called a complete synchronization (CS) and Anti-synchronization (AS), respectively.
- If $\forall pi = \pm 1$, then is called a Hybrid synchronization (HS).

3.1. Controllers for scaling factor $p_1 = 4, p_2 = 2, p_3 = 3, p_4 = 1$

If matrix P is selected as $P = diag(4, 2, 3, 1)$, i.e.:

$$e = \begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix} - \underbrace{\begin{bmatrix}4 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}_{P}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \quad (9)$$

Adding system (5) to system (6) yields an error system (9):

$$\begin{cases}\dot{e}_1 = be_2 - ae_1 - 2bx_2 + u_1\\ \dot{e}_2 = -ce_2 - d(e_4 - x_4) - e_3(e_1 + 4x_1) - x_3(3e_1 + 10x_1) + u_2\\ \dot{e}_3 = e_2(e_1 + 4x_1) + x_2(2e_1 + 5x_1) + 2k + u_3\\ \dot{e}_4 = e_1 + 3x_1 + u_4\end{cases} \quad (10)$$

Theorem 1: *A designed non-linear controller as*

$$\begin{cases}u_1 = -e_4 + x_2(2b - 2e_3) + e_2(3x_3 - e_3)\\ u_2 = -be_1 - dx_4 + 10x_1x_3\\ u_3 = -e_3 + e_1e_2 - 2k - 5x_1x_2\\ u_4 = -e_4 + de_2 - 3x_1\end{cases} \quad (11)$$

Then takes place Hybrid Synchronization between systems (5) and (6).

Proof: Inserting (11) in (10) we have:

$$\begin{cases}\dot{e}_1 = -ae_1 + e_2(b + 3x_3 - e_3) - 2e_3x_2 - e_4\\ \dot{e}_2 = -ce_2 - de_4 - e_3(e_1 + 4x_1) - e_1(3x_3 + b)\\ \dot{e}_3 = -e_3 + e_2(2e_1 + 4x_1) + 2e_1x_2\\ \dot{e}_4 = -e_4 + e_1 + de_2\end{cases} \quad (12)$$

A Lyapunov function is built as: $V(e_i) = e_i{}^T S e_i$

$$V(e_i) = [e_1 \quad e_2 \quad e_3 \quad e_4] \underbrace{\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}}_{S} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{13}$$

The time derivative of a preceding function $V(e_i)$ is given by:

$$\dot{V}(e_i) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 \tag{14}$$

$$\dot{V}(e_i) = -ae_1^2 - ce_2^2 - e_3^2 - e_4^2 \tag{15}$$

As a result, $\dot{V}(e_i)$ is negative definite (where $a = c = 10$), so the hybrid synchronization process has been achieved theoretically with control system (11).

3.2. Controllers for scaling factor $p_1 = 1, p_2 = -1, p_3 = 1, p_4 = -1$

If matrix *P* is selected as $P = diag(1, -1, 1, -1)$, i.e.:

$$e = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{P} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{16}$$

Adding system (5) to system (6) yields an error system (16):

$$\begin{cases} \dot{e}_1 = -ae_1 + be_2 - 2bx_2 + u_1 \\ \dot{e}_2 = -ce_2 - de_4 - e_3(e_1 + x_1) - x_3(e_1 + 2x_1) + u_2 \\ \dot{e}_3 = e_2(e_1 + x_1) - x_2(e_1 + 2x_1) + u_3 \\ \dot{e}_4 = e_1 + u_4 \end{cases} \tag{17}$$

Theorem 2: *A designed non-linear controller as*

$$\begin{cases} u_1 = -e_4 + x_2(2b + e_3) + e_2x_3 \\ u_2 = -be_1 + 2x_1x_3 - e_1e_3 \\ u_3 = -e_3 + e_1e_2 + 2x_1x_2 \\ u_4 = -e_4 + de_2 \end{cases} \tag{18}$$

Then takes place Hybrid Synchronization between systems (5) and (6).

Proof: Inserting (18) in (17) we have:

$$\begin{cases} \dot{e}_1 = -ae_1 + be_2 - e_4 + x_2e_3 + x_3e_2 \\ \dot{e}_2 = -ce_2 - de_4 - e_3(2e_1 + x_1) - x_3e_1 - be_1 \\ \dot{e}_3 = e_2(2e_1 + x_1) - x_2e_1 - e_3 \\ \dot{e}_4 = -e_4 + e_1 + de_2 \end{cases} \tag{19}$$

The time derivative (14) of a preceding function $V(e_i)$ is given by:

$$\dot{V}(e_i) = -ae_1^2 - ce_2^2 - e_3^2 - e_4^2 \tag{20}$$

As a result, $\dot{V}(e_i)$ is negative definite, so the hybrid synchronization process has been achieved theoretically with control system (18).

3.3. Numerical simulation

By applying the fourth-order Runge-Kutta scheme and for using a time step of 0.01 and the initial condition of both the drive system and the response system are following $(0.3, 0.2, 0.1, 0.2)$ and $(-0.1, 0.3, -0.4, 0.1)$ respectively, the differential equations of the controlled error dynamical system (10) for PHS and the controlled error dynamical system (17) for HS is adopted determined throughout simulation using MATLAB R2021.

- For the scaling factor $p_1 = 4, p_2 = 2, p_3 = 3, p_4 = 1$. The HPS for systems (5) and (6) through control (11) is shown in Figs. 5 and 6.
- For the scaling factor $p_1 = 1, p_2 = -1, p_3 = 1, p_4 = -1$. The HPS for systems (5) and (6) through control (18) is shown in Figs. 7 and 8.

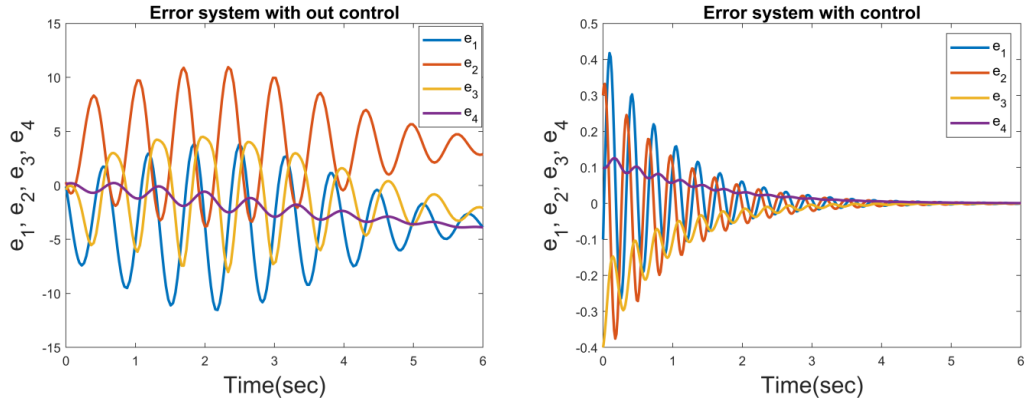

Figure 2. Hybrid Synchronization with and without error controllers (11).

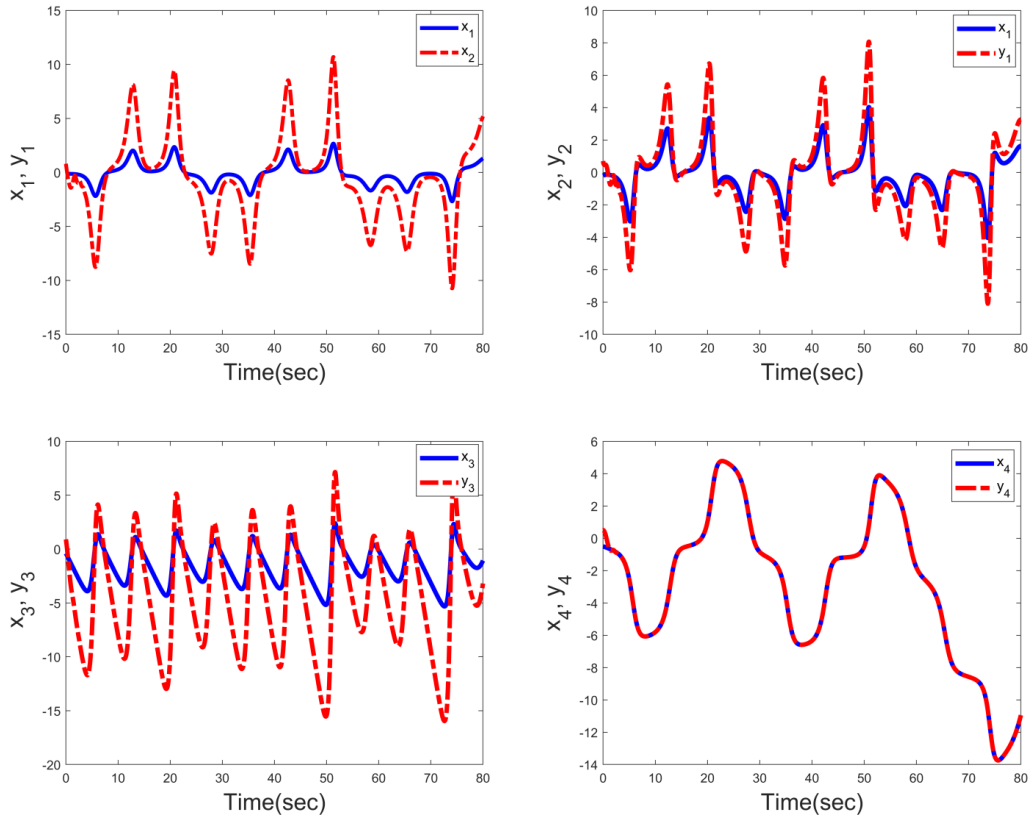

Figure 3. The HPS for the state variables with control (11) at scaling factors $p_1 = 4, p_2 = 2, p_3 = 3, p_4 = 1$

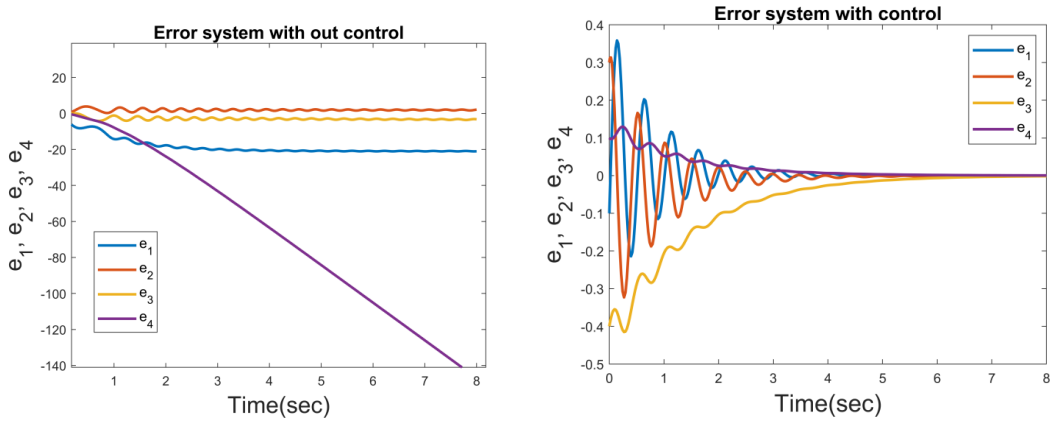

Figure 4. Hybrid Synchronization with and without error controllers (18).

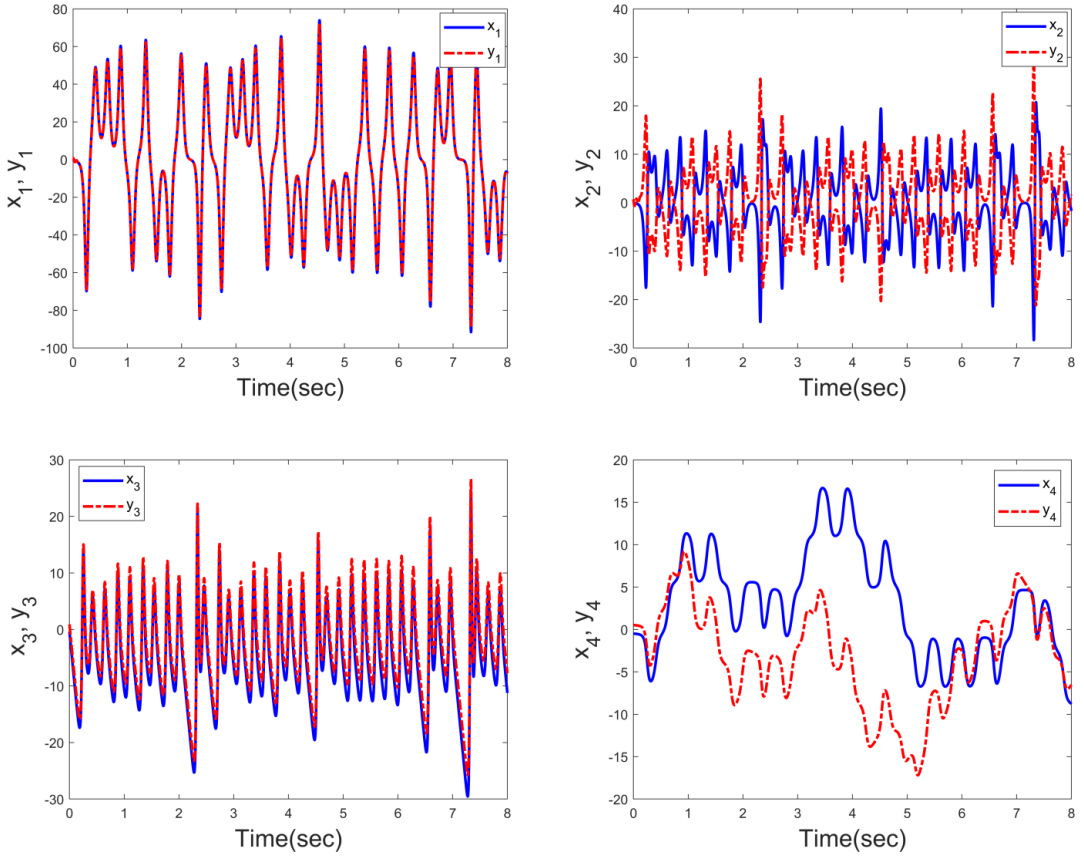

Figure 5. The HS for the state variables with control (18) at scaling factors $p_1 = 1, p_2 = -1, p_3 = 1, p_4 = -1$

4. CONCLUSION

The work's main goal was to explore hidden attractors and analyze complex nonlinear dissipative hyperchaotic systems. This was achieved by employing a state feedback method involving an eight-term system. Notably, the system features a maximal Lyapunov exponent (MLE) of 3.0743, surpassing the Lyapunov exponent of the original system. Various characteristics were explored, including the Lyapunov dimension, multistability, and hybrid projective synchronization (HPS), which is valuable across science, engineering, and technology. The system's potential applications span science, technology, neural networks, and robotics, with MATLAB V2021 simulations used to present the key findings..

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