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Stochastic Modelling on New Mixture Distribution with Properties and Their Application

M. Sakthivel[1], P. Pandiyan[2]
[1] Department of Statistics, Annamalai University
Chidambaram, Tamil Nadu-608002, India
[2] Department of Statistics, Annamalai University
Chidambaram, Tamil Nadu-608002, India

ABSTRACT: In this article, the authors propose a new two-parameter continuous distribution. It is called a new mixture distribution because it comes from the unique combination of exponential and gamma distributions (MEGD). Some statistical properties of the distribution are derived, such as the moments, moments generating function, reliability analysis. Also, the distribution of order statistics is presented, and entropies are derived. The application of the maximum likelihood estimation technique to the performance of traditional parameter estimates. Two authentic data sets describing cancer patients' survival are used to empirically demonstrate the potential importance and usability of the proposed distribution. Comparing the new mixture distribution to some other competing distribution, the analysis's results demonstrated that it performed quite well.

Keywords: Mixture Model, Moments, Order Statistics, Maximum Likelihood Estimation, Model Selection Technic AIC, BIC.

1. INTRODUCTION

Medical science is one of the most important applications of statistical analysis. On a great deal of occasions, specific statistical considerations are needed for cancer research in order to determine the model that best fits the survival data. The statistical distributions have been extensively utilized for analyzing time-to-event data, also referred to as survival or reliability data, in different areas of applicability, including medical science. In recent years, an impressive set of new statistical distributions has been explored by statisticians. The necessity of developing an extended class of classical distribution is analysis, biomedicine, reliability, insurance, and finance. Recently, many researchers have been working on this area and have proposed new methods to develop improved probability distributions with utility. [11] shanker distribution proposed as a mixture of exponential (θ) and gamma (2, θ) distribution. [10] the combination of exponential (θ), gamma (2, θ) and gamma (4, θ) with respectively mixing proportion $\frac{\theta^3}{\theta^3+\theta^2+6}, \frac{\theta^2}{\theta^3+\theta^2+6}\ and\ \frac{6}{\theta^3+\theta^2+6}$ propose Uma distribution.[15] Akash distribution is atwo component mixture of an exponential distribution and gamma distribution with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$. [12] Aradhana distribution is a three-component mixture of an exponential distribution and gamma with mixing proportions $\frac{\theta^2}{\theta^2+2\theta+2}, \frac{2\theta}{\theta^2+2\theta+2}$and $\frac{2}{\theta^2+2\theta+2}$. [19] Komal distribution with properties and application in survival analysis. [16] Garima distribution and its application model behavioral science data.

In order to create the new mixed distribution with parameters λ and θ, which we will refer to as mix (λ, θ), this study will combine the gamma distribution with parameters λ=3 and θ, gamma (3, θ), and the exponential distribution with parameter θ, Exp(θ) with mixing proportions $\frac{\lambda}{\lambda+1}\ and\ \frac{1}{\lambda+1}$.

If random variable X has the probability density function (pdf) $f(x) = \sum_{i=1}^{n} \omega_i f_i(x)$, then it is said to have a mixture of two distribution $f_1(x), \dots, f_n(x)$. The missing weight where $0 \le x_i \le 1$ so that $\sum_{i=1}^{n} \omega_i f_i(x) = 1$.

An exponential distribution random variable X with a parameter $\theta > 0$ is described by its pdf as

$$f(x) = \theta e^{-\theta x} \quad , x > 0, \theta > 0$$

Given the gamma distribution with parameters $\lambda = 3, \theta$ if the pdf is obtained as

$$f(x) = \frac{1}{2}\theta^3 x^2\ e^{-\theta x} \quad , x > 0, \theta > 0$$

The present study examines, through an analysis of several cancer data sets, the adaptability of the proposed distribution to represent the survival time. A further goal is to use the maximum likelihood method to estimate the unknown model parameters.

The remainder of the work is structured around a newly defined mixture distribution (MGED), its special instances, a use-full extension for its pdf and cdf, and some demonstrations of the proposed distribution's features. The distribution parameters' maximum likelihood estimators (MLEs) are found. Various simulation studies are conducted to evaluate the MLEs' performance. Lastly, various uses of the newly developed mixture distribution MGED fitting datasets are demonstrated in comparision to other well-known classical distributions. Throughout this research, the statistical programming language R was used for all computations.

2. NEW MIXTURE DISTRIBUTION

They introduce a new mixture probability distribution to the life time model. A mixture of exponential and gamma distribution (MEGD)is a well-known distribution and importance in the study of growth, lifetime data. A continuous random variable X has a mixture distribution, if its pdf $f(x;\lambda,\theta)$and cdf $F(x;\lambda,\theta)$ are, given by

$$f(x;\lambda,\theta)=\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x} \qquad ,x>0,\theta>0,\lambda>0 \tag{1}$$

The function defined in (1) represents a probability distribution function pdf of the new distribution as a mixture of exponential and gamma (MEGD)$f(x;\lambda,\theta)\ for\ all\ x>0$

$$\int_0^\infty f(x;\lambda,\theta)\,dx=\int_0^\infty \frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}dx$$

$$=\frac{\theta}{\lambda+1}\left(\frac{\lambda+1}{\theta}\right)$$

$$=1$$

The cumulative distribution function cdf is obtained as

$$F(x;\lambda,\theta)=\int_0^x \frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2t^2+1\right)e^{-\theta t}dt$$

Then, the cumulative distribution function cdf of the new mixture of exponential and gamma distribution (MEGD)are obtained as

$$F(x;\lambda,\theta)=1-\left(1+\frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2}+1\right)\right)e^{-\theta x} \qquad x>0,\theta>0,\lambda>0 \tag{2}$$

As a cumulative distribution function, the function satisfies the necessary conditions

$$\lim_{x\to 0}F(x;\lambda,\theta)=\lim_{x\to 0}\left(1-\left(1+\frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2}+1\right)\right)e^{-\theta x}\right)$$

$$=1-1=0$$

$$\lim_{x\to\infty}F(x;\lambda,\theta)=\lim_{x\to\infty}\left(1-\left(1+\frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2}+1\right)\right)e^{-\theta x}\right)$$

$$=1-0=1$$

3. RELIABILITY ANALYSIS

This section will provide the reliability function, hazard function, reverse hazard function, cumulative hazard function, odds rate, Mills ratio, and mean residual function for the specified new mixture distribution (MGED).

3.1 Survival function

The survival function of new mixture distribution (MEGD)is obtained as

$$S(x)=1-F(x;\lambda,\theta)$$

$$S(x)=\left(1+\frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2}+1\right)\right)e^{-\theta x}$$

3.2 Hazard rate function

An important metric for describing life phenomena is the hazard rate function of the new mixture distribution (MGED), which is defined by $h(x)=\frac{f(x;\lambda,\theta)}{\mathbf{1}-F(x;\lambda,\theta)}$.

$$h(x) = \left(\frac{\left(\frac{1}{2}\lambda\theta^2 x^2 + 1\right)\theta}{(\lambda+1) + \lambda\theta x\left(\frac{\theta x}{2} + 1\right)} \right)$$

3.3 Revers hazard rate

The Revers hazard rate of new mixture distribution (MEGD)is obtained as

$$h_r(x) = \frac{f(x;\lambda,\theta)}{F(x;\lambda,\theta)}$$

$$h_r(x) = -\left(\frac{\left(\frac{1}{2}\lambda\theta^2 x^2 + 1\right)}{\lambda x\left(\frac{\theta x}{2} + 1\right)} \right)$$

3.4 Cumulative hazard function

The Cumulative hazard function of new mixture distribution (MEGD)is obtained as

$$H(x) = -\ln(1 - F(x;\lambda,\theta))$$

$$H(x) = -\ln\left(\left(1 + \frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2} + 1\right)\right) e^{-\theta x} \right)$$

3.5 Odds rate function

The Odds rate function of new mixture distribution (MEGD)is obtained as

$$O(x) = \frac{F(x;\lambda,\theta)}{1 - F(x;\lambda,\theta)}$$

$$O(x) = \left(\frac{1 - \left(1 + \frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2} + 1\right)\right)}{\left(1 + \frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2} + 1\right)\right)} \right)$$

3.6 Mean residual function

The mean residual function of new mixture distribution (MEGD)is obtained as

$$M(x) = \frac{1}{S(x)} \int_x^\infty t\, f(t) dt - x$$

$$M(x) = \frac{1}{\left(1 + \frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2} + 1\right)\right) e^{-\theta x}} \int_x^\infty t\, \frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2 t^2 + 1\right) e^{-\theta t} dt - x$$

Then, the mean residual function can be obtained as

By letting, $u = t^3,\ du = 3t^2\, dt \quad and \quad dv = e^{-\theta t}, v = -\frac{e^{-\theta t}}{\theta}$

Then, solving the integral, we get

$$M(x) = \left(\frac{\left(\lambda\theta x^2\,(\theta x + 3) + x(3\lambda + 1) + \frac{(3\lambda+1)}{\theta}\right)}{(\lambda+1)\left(1 + \frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2} + 1\right)\right)} \right) - x$$

4. STATISTICAL PROPERTIES

In this section, to derived the structural properties, moments, the moment generating function, Characteristics function and r^{th} moment for the new mixture distribution of the random variable is also derived. Among the measures examined were the mean, variance, coefficient of variation, skewness, kurtosis, and dispersion.

4.1 Moments

If a random X has the pdf given by equation (1), then the corresponding r^{th} moments is given by

$$E(X^r) = \mu_r' = \int_0^\infty x^r\ f(x;\lambda,\theta)\, dx$$

$$\mu_r' = \int_0^\infty x^r \, \frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}\,dx$$

Solving the integration, using the following gamma function

$$\int_0^\infty x^{z-1}e^{-px}\,dx = \frac{\Gamma(z)}{p^z}$$

$$\mu_r' = \left(\frac{\lambda\Gamma(r+3)}{2\theta^r(\lambda+1)}+\frac{\Gamma(r+1)}{\theta^r(\lambda+1)}\right) \quad (3)$$

Where $\Gamma(.)$ Is the gamma function. Subsequently, the mean and variance can be obtained by substituting $r=$ 1,2,3,4 in equation (3)

$$E(X) = \left(\frac{3\lambda+1}{\theta(\lambda+1)}\right)$$

$$E(X^2) = \left(\frac{2(6\lambda+1)}{\theta^2(\lambda+1)}\right)$$

$$E(X^3) = \left(\frac{6(10\lambda+1)}{\theta^3(\lambda+1)}\right)$$

$$E(X^4) = \left(\frac{24(15\lambda+1)}{\theta^4(\lambda+1)}\right)$$

Variance $= \sigma^2 = E(X^2) - \left(E(X)\right)^2$

$$\sigma^2 = \frac{2(\lambda 6+1)}{\theta^2(\lambda+1)} - \left(\frac{3\lambda+1}{\theta(\lambda+1)}\right)^2$$

After simplification we get,

$$\sigma^2 = \left(\frac{3\lambda^2+8\lambda+1}{\left(\theta(\lambda+1)\right)^2}\right)$$

$$\sigma = \left(\frac{\sqrt{3\lambda^2+8\lambda+1}}{\left(\theta(\lambda+1)\right)}\right)$$

Coefficient of Variation

$$C.V\left(\frac{\sigma}{\mu}\right) = \frac{\sqrt{3\lambda^2+8\lambda+1}}{3\lambda+1}$$

Skewness

$$Sk(X) = \frac{E(X^3)-3\mu E(X^2)+2\mu^3}{\sigma^3}$$

After simplification we get,

$$Sk(X) = \left(\frac{12\lambda^2+86\lambda+14}{\theta^3(\lambda+1)^3}\right)$$

Kurtosis

$$Ku(X) = \frac{E(X^4)-4\mu E(X^3)+6\mu^2E(X^2)-3\mu^4}{\sigma^4}$$

$$Ku(X) = \left(\frac{-(576\lambda^2+1200\lambda+24+\lambda^2\theta 720+312\lambda\theta+24\theta)}{24(15\lambda+1)(\lambda+1)}\right)$$

Dispersion

Dispersion= $\frac{\sigma^2}{\mu}$

$$= \left(\frac{3\lambda^2+8\lambda+1}{\theta(\lambda+1)(3\lambda+1)}\right)$$

4.2 Moment generating function

If a random variable X has the pdf by (1), then the corresponding r^{th} moments is given by.

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x;\lambda,\theta)\ dx$$
$$M_X(t) = \frac{\theta}{\lambda+1}\int_0^\infty e^{tx}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}\ dx$$
$$M_X(t) = \frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2\int_0^\infty x^2e^{-(\theta-t)x}\,dx + \int_0^\infty e^{-(\theta-t)x}\,dx\right)$$

Then, the rth moments can be obtained as

By assuming, $u = x^2 \quad and \quad dv = -\frac{e^{-(\theta-t)x}}{(\theta-t)}$

$$M_X(t) = \frac{\theta}{(\lambda+1)(\theta-t)}\left(\frac{2}{(\theta-t)^2}+1\right) \tag{4}$$

4.3 Charateristics function

The characteristics function of a random variable X can be defined with form

$$\phi_X(t) = E(e^{itx}) = \int_0^\infty e^{itx} f(x;\lambda,\theta)\ dx$$
$$\phi_X(t) = M_X(it)$$

the characteristics function of a random variable X whose pdf in equation (1) can be obtained similarly as

$$\phi_X(t) = \frac{\theta}{(\lambda+1)(\theta-it)}\left(\frac{2}{(\theta-it)^2}+1\right) \tag{5}$$

5. HORMONIC MEAN

The Hormonic mean of the new mixture distribution (MEGD) is defined as

$$H.M = \int_0^\infty \frac{1}{x} f(x;\lambda,\theta)dx$$
$$H.M = \int_0^\infty \frac{1}{x}\,\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}dx$$
$$H.M = \frac{\theta}{\lambda+1}\int_0^\infty \frac{1}{x}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}dx$$

Solving the integration, using the following gamma function

$$H.M = \frac{1}{(\lambda+1)}\left(\frac{\lambda\theta}{2}-1\right)$$

6. MEAN DEVIATION

Let X be a random variable from new mixture distribution (MEGD) with mean μ. Then the deviation from mean is defined as

$$D(\mu) = E(|X-\mu|)$$
$$D(\mu) = \int_0^\infty |X-\mu|\ \ f(x;\lambda,\theta)dx$$
$$D(\mu) = \int_0^\mu (\mu-x)\,f(x;\lambda,\theta)dx + \int_\mu^\infty (x-\mu)\ f(x;\lambda,\theta)dx$$
$$D(\mu) = \mu\int_0^\mu f(x;\lambda,\theta)dx - \int_0^\mu x\ f(x;\lambda,\theta)dx + \int_\mu^\infty x\ f(x;\lambda,\theta)dx - \int_\mu^\infty \mu\ f(x;\lambda,\theta)dx$$
$$D(\mu) = \mu F(\mu) - \int_0^\mu x\ f(x;\lambda,\theta)dx - \mu[1-F(\mu)] + \int_\mu^\infty x\ f(x;\lambda,\theta)dx$$
$$D(\mu) = 2\mu F(\mu) - 2\int_0^\mu x\ f(x;\lambda,\theta)dx$$

Then,

$$\int_0^\mu x\,f(x;\lambda,\theta)\,dx = \int_0^\mu x\,\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}\,dx$$

Then, the deviation from mean can be obtained as

By letting, $u = x^3 \quad and \quad dv = e^{-\theta x} \quad then \ \ du = 3x^2dx$

Then, solving the integral to simplifying this, we get

$$D(\mu)=\left\{\begin{array}{c} 2\left(\frac{3\lambda+1}{\theta(\lambda+1)}\right)\left(1-\left(1+\frac{\lambda\theta\mu}{(\lambda+1)}\left(\frac{\theta\mu}{2}+1\right)\right)e^{-\theta\mu}\right) \\ +\left(\frac{2}{\lambda+1}\left(\left(\frac{3\lambda}{\theta}+1\right)\left(1-\ e^{-\theta\mu}\right)-2\mu\left(\frac{\lambda\theta\mu}{2}(\theta\mu+3)+(3\lambda+1)\right)\ e^{-\theta\mu}\right)\right)\end{array}\right\}$$

7. MEDIAN

Let X be a random variable from new mixture distribution (MEGD) with median M. Then the mean deviation from median is defined as

$$D(M)=E(|X-M|)$$

$$D(M)=\int_0^{\infty}|X-M|\ f(x)dx$$

$$D(M)=\int_0^{M}(M-x)f(x)dx+\int_M^{\infty}(x-M)f(x)dx$$

$$D(M)=MF(M)-\int_0^{M}x\,f(x)dx-M[1-F(M)]+\int_M^{\infty}x\,f(x)dx$$

$$D(M)=\mu-2\int_0^{M}x\,f(x)dx$$

Now,

$$\int_0^{M}x\,f(x)dx=\int_0^{M}x\,\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}\,dx$$

Then, the deviation from median can be obtained as

Let assuming, $u=x^3\quad and\quad dv=e^{-\theta x}\quad then\ \ du=3x^2dx$

Then, solving the integral, we get

$$D(M)=\left\{\frac{3\lambda+1}{\theta(\lambda+1)}-\left(\frac{2}{\lambda+1}\left(\left(\frac{3\lambda}{\theta}+1\right)\left(1-\ e^{-\theta M}\right)-2p\left(\frac{\lambda\theta M}{2}(\theta M+3)+(3\lambda+1)\right)e^{-\theta M}\right)\right)\right\}$$

8. ORDER STATISTICS

The derived pdf of the i^{th} order statistics of the new mixture distribution (MEGD). Let $X_1,X_2,\dots,X_n$ be a simple random sample from new mixture distribution with cdf and pdf given by (1) and (2), respectively. Let $X_{(1:n)}\le X_{(2:n)}\le\cdots\le X_{(n:n)}$ denote the order statistics obtained from this sample. We now given the pdf of $X_{r:n}$, say $f_{r;n}(x)$ of $X_{r:n}$, $i=1,2,\dots,n$. The pdf of the r^{th} order statistics $X_{r:n}$, $r=1,2,\dots,n$ is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!\,(n-r)!}\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}f(x),x>0 \qquad (6)$$

Where $F(.)$ and $f(.)$ are given by (1) and (2) respectively,

$$and\quad C_{r:n}=\frac{n!}{(r-1)!\,(n-r)!}$$

$$f_{r:n}=C_{r:n}\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}f(x)$$

Then, using the following binomial series expansion

$$(1-z)^a=\sum_{j=0}^{\infty}(-1)^j\binom{a}{j}\ z^j$$

$$f_{r:n}=C_{r:n}\sum_{s=0}^{\infty}(-1)^s\binom{n-r}{s}\left(F(x)\right)^{r+s+1}f(x)$$

$$f_{r:n}=\left\{C_{r:n}\sum_{s=0}^{\infty}(-1)^s\binom{n-r}{s}\left(1-\left[1+\frac{\lambda\theta x}{(\lambda+1)}\left(\frac{\theta x}{2}+1\right)\right]e^{-\theta x}\right)^{r+s+1}\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}\right\}$$

After simplification we get,

$$f_{r:n}=\left\{\begin{matrix}C_{r:n}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{q+s}\binom{n-r}{s}\binom{r+s-1}{q}\binom{q}{p}\binom{p}{t}\left(\frac{\theta x}{2}\right)^{t}\left(\frac{\lambda\theta x}{(\lambda+1)}\right)^{p}e^{-\theta(q+1)x}\\ \times\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^{2}x^{2}+1\right)\end{matrix}\right\}$$

First order statistics

$$f_{1:n}=\left\{C_{1:n}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{q+s}\binom{n-1}{s}\binom{s}{p}\binom{p}{q}\binom{q}{t}\left(\frac{\theta x}{2}\right)^{t}\left(\frac{\lambda\theta x}{(\lambda+1)}\right)^{p}e^{-\theta(q+1)x}\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^{2}x^{2}+1\right)\right\}$$

nth order statistics

$$f_{n:n}=\left\{C_{n:n}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{q+s}\binom{n+s-1}{q}\binom{q}{p}\binom{p}{t}\left(\frac{\theta x}{2}\right)^{t}\left(\frac{\lambda\theta x}{(\lambda+1)}\right)^{p}e^{-\theta(q+1)x}\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^{2}x^{2}+1\right)\right\}$$

8.1 Quantial function

The quantile function of a distribution with cdf, $F(x;\lambda,\theta)$, is defined by $q=F\left(x_q;\lambda,\theta\right)$, where $0<q<1$. Thus, the quantile function of new mixture distribution (MEGD) is given by

$$1-q=\left(1+\frac{\lambda\theta x_q}{(\lambda+1)}\left(\frac{\theta x_q}{2}+1\right)\right)e^{-\theta x_q}$$

Figure 8 shows the quantile plot for different values of $\theta\ and\ \lambda$.

9. ENTROPIES

In this section, derived the Rényi entropy, and Tsallis entropy from the distribution.
It is well known that entropy and information can be considered measures of uncertainty or the randomness of a probability distribution. It is applied in many fields, such as engineering, finance, information theory, and biomedicine. The entropy functionals for probability distribution were derived on the basis of a variational definition of uncertainty measure.

9.1 Rényi entropy

Entropy is defined as a random variable X is a measure of variation of the uncertainty. It is used in many fields, such as engineering, statistical mechanics, finance, information theory, biomedicine, and economics. The entropy measure is the Rényi of order which is defined as

$$R_\delta=\frac{1}{1-\delta}\log\int_0^\infty[f(x;\lambda,\theta)]^\delta\ dx\qquad;\delta>0,\delta\neq1$$

$$R_\delta=\frac{1}{1-\delta}\log\left(\frac{\theta}{\lambda+1}\right)^\delta\int_0^\infty\left(\frac{1}{2}\lambda\theta^2x^2+1\right)^\delta e^{-\delta\theta x}\ dx$$

Then, the integration, using the following binomial series expansion

$$(1+x)^a=\sum_{j=0}^{n}\binom{a}{j}\ (x)^j$$

Then, the following power series expansion

$$a^x=\sum_{k=0}^{\infty}\frac{(x\ln a)^k}{k!}$$

$$R_\delta=\frac{1}{1-\delta}\log\left\{\left(\frac{\theta}{\lambda+1}\right)^\delta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^i\binom{\delta}{i}\frac{(i\ln\lambda)^j}{j!}\frac{(2i\ln\theta)^k}{k!}\int_0^\infty x^{(2i+1)}\ e^{-\delta\theta x}\,dx\right\}$$

Solving the integration, using the following gamma function

$$\frac{\Gamma(z)}{a^z}=\int_0^\infty x^{z-1}\ e^{-ax}\,dx$$

$$R_\delta=\frac{1}{1-\delta}\log\left\{\left(\frac{\theta}{\lambda+1}\right)^\delta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^i\binom{\delta}{i}\frac{(i\ln\lambda)^j}{j!}\frac{(2i\ln\theta)^k}{k!}\frac{\Gamma(2i+1)}{(\delta\theta)^{2i+1}}\right\}$$

9.2 Tsallis entropy

The Boltzmann-Gibbs (B-G) statistical properties initiated by Tsallis have received a great deal of attention. This generalization of (B-G) statistics was first proposed by introducing the mathematical expression of Tsallis entropy (Tsallis, (1988) for continuous random variables, which is defined as

$$T_\delta = \frac{1}{\delta-1}\left(1-\int_0^\infty [\,f(x;\lambda,\theta)]^\delta\ dx\right) \qquad ;\delta>0,\delta\neq 1$$

$$T_\delta = \frac{1}{\delta-1}\left\{1-\left(\int_0^\infty \left(\frac{\theta}{\lambda+1}\right)^\delta \left(\ \left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}\right)^\delta\ dx\right)\right\}$$

Then, solving the integration to simplifying this, we get

$$T_\delta = \frac{1}{\delta-1}\left\{1-\left(\left(\frac{\theta}{\lambda+1}\right)^\delta \sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty \left(\frac{1}{2}\right)^i \binom{\delta}{i}\frac{(i\ln\lambda)^j}{j!}\frac{(2i\ln\theta)^k}{k!}\frac{\Gamma(2i+1)}{(\delta\theta)^{2i+1}}\right)\right\}$$

10. STOCHASTIC ORDERING

Stochastic ordering is an important technique in finance and dependability for assessing the relative performance of the models. Let X and Y be two random variables with pdf, cdf, and reliability functions $f(x), f(y), F(x), F(y). S(x) = 1 - F(x) and\ F(y)$.

1- Likelihood ratio order ($X \leq_{LR}\ Y$) if $\frac{f_X(x;\lambda,\theta)}{f_Y(x;\lambda,\theta)}$ decreases in x
2- Stochastic order ($X \leq_{ST}\ Y$) if $F_X(x;\lambda,\theta) \geq F_Y(x;\lambda,\theta)$ for all x
3- Hazard rate order ($X \leq_{HR}\ Y$) if $h_X(x;\lambda,\theta) \geq F_Y(x;\lambda,\theta)$ for all x
4- Mean residual life order ($X \leq_{MRL}\ Y$) if $MRL_X(X) \leq MRL_Y(X)$ for all x

Prove that the new mixture distribution (MEGD) complies with the ordering with the highest likelihood (the likelihood ratio ordering).

Assume that X and Y are two independent Random variables with probability distribution function$f_x(x;\lambda,\theta)$and $f_y(x;\alpha,\beta)$. If $\lambda < \alpha$ and $\theta < \beta$, then

$$\Lambda = \frac{f_x(x;\lambda,\theta)}{f_y(x;\alpha,\beta)}$$

$$\Lambda = \left(\frac{\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2x^2+1\right)e^{-\theta x}}{\frac{\beta}{\alpha+1}\left(\frac{1}{2}\alpha\beta^2x^2+1\right)e^{-\beta x}}\right)$$

$$\Lambda = \left(\frac{\theta(\beta+1)\left(\frac{1}{2}\lambda\theta^2x^2+1\right)}{\alpha(\lambda+1)\left(\frac{1}{2}\alpha\beta^2x^2+1\right)}\right)e^{(\alpha-\theta)x}$$

Therefore, the log-likelihood function is given by

$$\log[\Lambda] = \log\left(\frac{\theta(\beta+1)}{\alpha(\lambda+1)}\right) + \log\left(\frac{1}{2}\lambda\theta^2x^2+1\right) - \log\left(\frac{1}{2}\alpha\beta^2x^2+1\right) - (\alpha-\theta)x$$

Differentiating with respect to x, we get.

$$\frac{\partial \log[\Lambda]}{\partial x} = \left(\frac{\lambda\theta^2x}{\frac{1}{2}\lambda\theta^2x^2+1}\right) - \left(\frac{\alpha\beta^2x}{\frac{1}{2}\alpha\beta^2x^2+1}\right) + (\alpha-\theta) = 0$$

Hence, $\frac{\partial \log[\Lambda]}{\partial x} < 0$ if $\lambda < \alpha, \theta < \beta$.

11. BONFERRONI AND LORENZ CURVES

The Bonferroni and Lorenz curves have been obtained using the new mixture distribution (MEGD) in this section.

The Bonferroni and Lorenz curve is a powerful tool in the analysis of distributions and has applications in many fields, such as economies, insurance, income, reliability, and medicine. The Bonferroni and Lorenz cures for a X be the random variable of a unit and $f(x)$ be the probability density function of x. $f(x)dx$ will be represented by the probability that a unit selected at random is defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x\, f(x;\lambda,\theta)dx \quad and$$

$$L(p) = \frac{1}{\mu}\int_0^q x\, f(x;\lambda,\theta)dx$$

Where, $q = F^{-1}(p);\quad q\epsilon[0,1]$ and

$$\mu = E(X) = \left(\frac{3\lambda + 1}{\theta(\lambda + 1)}\right)$$

Hence the Bonferroni and Lorenz curves of our distribution are, given by

$$B(p) = \frac{1}{p\mu}\int_0^p x\ \frac{\theta}{\lambda + 1}\left(\frac{1}{2}\lambda\theta^2 x^2 + 1\right)e^{-\theta x}\,dx$$

$$B(p) = \frac{1}{p\left(\frac{3\lambda + 1}{\theta(\lambda + 1)}\right)}.\frac{\theta}{\lambda + 1}\left(\frac{1}{2}\lambda\theta^2\int_0^p x^3 e^{-\theta x}\,dx + \int_0^p x\,e^{-\theta x}\,dx\right)$$

Let assuming, $u = x^3$, $dv = e^{-\theta x}\,dx,\quad then\quad du = 3x^2\,dx,\ \ v = -\frac{1}{\theta}e^{-\theta x}$

Then, solving the integration to simplifying this, we get

$$B(p) = \left\{\frac{\theta}{p(3\lambda + 1)}\left(\left(\frac{3\lambda}{\theta} + 1\right)\left(1 - e^{-\theta p}\right) - p\left(\frac{\lambda\theta p}{2}(\theta p + 3) + (3\lambda + 1)\right)\ e^{-\theta p}\right)\right\}$$

$$L(p) = pB(p)$$

$$L(p) = \left\{\frac{\theta}{(3\lambda + 1)}\left(\left(\frac{3\lambda}{\theta} + 1\right)\left(1 - e^{-\theta p}\right) - p\left(\frac{\lambda\theta p}{2}(\theta p + 3) + (3\lambda + 1)\right)\ e^{-\theta p}\right)\right\}$$

12. LEAST SQUARE AND WEIGHTED LEAST SQUARE ESTIMATIONS

The third approach for estimating the model parameters is covered in this section. It involves estimating the weighted least square (WLSE) and ordinary least square (OLSE).

$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are the order statistics of a random sample derived from a probability distribution. The mean and the variance are defined for the i^{th} order statistic.

$$E\left(F\left(x_{(i)};\lambda,\theta\right)\right) = \frac{i}{n + 1}\quad and$$

$$var\left(F\left(x_{(i)};\lambda,\theta\right)\right) = \frac{i(n - i + 1)}{(n + 1)^2(n + 2)}\qquad for\ all\ i = 1,2,\dots,n \tag{7}$$

Swain et al. (1988) introduced OLS and WLS [4]. By minimizing the function that follows with regard to the parameter, they can obtain OLS estimates for the parameters.

$$R = \sum_{i=1}^{n}\left(F\left(x_{(i)};\lambda,\theta\right) - \mathcal{F}(i)\right)^2 \tag{8}$$

In the case of the distribution under consideration, the theoretical CDF of the observation $x_{(i)}$ is represented by $F\left(x_{(i)};\lambda,\theta\right)$, while the empirical CDF, $\mathcal{F}(i)$ which is often calculated by $\mathcal{F}(i) = \frac{i}{n+1}$

$$R = \sum_{i=1}^{n}\left(F\left(x_{(i)};\lambda,\theta\right) - \frac{i}{n + 1}\right)^2 \tag{9}$$

In substituting its cdf specified in Eq. (2), for $\mathcal{F}(i)$ in the previous equation, this function can be produced for the new mixture distribution (MEGD) as follows.

$$R(\lambda,\theta) = \sum_{i=1}^{n}\left\{\left(\frac{\lambda\theta x_{(i)}}{(\lambda + 1)}\left(\frac{\theta x_{(i)}}{2} + 1\right)\right)e^{-\theta x_{(i)}} - \frac{i}{n + 1}\right\}^2 \tag{10}$$

The following equation must be solved in order to find the OLS estimates: minimize in Eq. (10), with respect to the parameters.

$$\frac{S(\lambda,\theta)}{\partial\theta} = \sum_{i=1}^{n}\zeta_i^{(1)}(\lambda,\theta)\left\{\left(\frac{\lambda\theta x_{(i)}}{(\lambda + 1)}\left(\frac{\theta x_{(i)}}{2} + 1\right)\right)e^{-\theta x_{(i)}} - \frac{n + 1 - i}{n + 1}\right\} = 0$$

$$\frac{S(\lambda,\theta)}{\partial\lambda} = \sum_{i=1}^{n}\zeta_i^{(2)}(\lambda,\theta)\left\{\left(\frac{\lambda\theta x_{(i)}}{(\lambda + 1)}\left(\frac{\theta x_{(i)}}{2} + 1\right)\right)e^{-\theta x_{(i)}} - \frac{n + 1 - i}{n + 1}\right\}x_{(i)} = 0$$

$$\zeta_i^{(1)}(\lambda,\theta) = \left(\frac{x_{(i)}\left(\theta - \lambda\theta + \lambda\theta x_{(i)}(\theta - 1)\right)}{2(\lambda + 1)^2}\right)e^{-\theta x_{(i)}} \tag{11}$$

$$\zeta_i^{(2)}(\lambda,\theta) = \left(\frac{x_{(i)}\left(\theta x_{(i)} + 2(\lambda + 1) - \lambda\theta x_{(i)}\right)}{2(\lambda + 1)^2}\right)e^{-\theta x_{(i)}} \tag{12}$$

By minimizing the following with regard to the parameters, we can derive WLS estimates for the parameters.

$$W=\sum_{i=1}^{n}\omega_i\left(F\left(x_{(i)};\lambda,\theta\right)-\frac{i}{n+1}\right)^2 \qquad \omega_i=\frac{(n+1)^2(n+2)}{i(n+1-i)} \tag{13}$$

After substituting its cdf described in Eq. (2), for $F\left(x_{(i)};\lambda,\theta\right)$ in the preceding equation, this function can be produced for the new mixture distribution (MEGD).

$$W(\lambda,\theta)=(n+1)^2(n+2)\sum_{i=1}^{n}\frac{\left(\left(\frac{\lambda\theta x_{(i)}}{(\lambda+1)}\left(\frac{\theta x_{(i)}}{2}+1\right)\right)e^{-\theta x_{(i)}}-\frac{n+1-i}{n+1}\right)^2}{i(n+1-i)} \tag{14}$$

The following equation can be solved to obtain the WLS estimates by minimizing in Eq. (14), with respect to the parameters.

$$\frac{W(\lambda,\theta)}{\partial\theta}=\sum_{i=1}^{n}\left\{\left(\frac{\lambda\theta x_{(i)}}{(\lambda+1)}\left(\frac{\theta x_{(i)}}{2}+1\right)\right)e^{-\theta x_{(i)}}-\frac{n+1-i}{n+1}\right\}\frac{\left(\frac{\lambda\theta^2x_{(i)}^2}{2(\lambda+1)}+\frac{\lambda\theta x_{(i)}}{(\lambda+1)}\right)e^{-\theta x_{(i)}}}{i(n+1-i)}=0$$

$$\frac{W(\lambda,\theta)}{\partial\lambda}=\sum_{i=1}^{n}\left\{\left(\frac{\lambda\theta x_{(i)}}{(\lambda+1)}\left(\frac{\theta x_{(i)}}{2}+1\right)\right)e^{-\theta x_{(i)}}-\frac{n+1-i}{n+1}\right\}\frac{x_{(i)}e^{-\theta x_{(i)}}}{i(n+1-i)}=0$$

13. MAXIMUM PRODUCT OF SPACING ESTIMATION

The MPS was developed in 1970 by Chening and Amin [7]. maximize the following function is the concept behind this approach.

$$P=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i \tag{15}$$

Where $D_i=F\left(x_{(i)}-F\left(x_{(i-1)}\right)\right)$ $\quad$ *and* $F\left(x_{(0)}\right)=0,\ \ F\left(x_{(n+1)}\right)=1$

By substituting its cdf specified in Eq. (2), for $F\left(x_{(i)}\right)$ in the previous equation, this function can be produced for the new mixture distribution (MEGD).

$$P(\lambda,\theta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log(Q_i)-\log\left((\lambda+1)\right) \tag{16}$$

$$Q_i=\lambda\theta^2\left\{x_{(i-1)}^2e^{-\theta x_{(i-1)}}-x_{(i)}^2e^{-\theta x_{(i)}}\right\}+\lambda\theta^2\left\{x_{(i-1)}e^{-\theta x_{(i-1)}}-x_{(i)}e^{-\theta x_{(i)}}\right\}+\left\{e^{-\theta x_{(i-1)}}-e^{-\theta x_{(i)}}\right\}$$

By solving the following equation and maximizing in Eq. (16), we may determine the MPS estimations.

$$\frac{P(\lambda,\theta)}{\partial\theta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{Q_i}\frac{\partial Q_i}{\partial\theta}=0$$

$$\frac{P(\lambda,\theta)}{\partial\lambda}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{Q_i}\frac{\partial Q_i}{\partial\lambda}-\frac{1}{(\lambda+1)}=0$$

where

$$\frac{\partial Q_i}{\partial\theta}=\lambda\theta\left\{x_{(i-1)}^2e^{-\theta x_{(i-1)}}-x_{(i)}^2e^{-\theta x_{(i)}}\right\}+\lambda\left\{x_{(i-1)}e^{-\theta x_{(i-1)}}-x_{(i)}e^{-\theta x_{(i)}}\right\}$$

$$\frac{\partial Q_i}{\partial\lambda}=\theta^2\left\{x_{(i-1)}^2e^{-\theta x_{(i-1)}}-x_{(i)}^2e^{-\theta x_{(i)}}\right\}+\theta\left\{x_{(i-1)}e^{-\theta x_{(i-1)}}-x_{(i)}e^{-\theta x_{(i)}}\right\}$$

14. CRAMER-VON-MISES METHOD

In 1971, MacDonald proposed the Cramer-von-Mises technique [6]. Reduce the function is the concept behind this approach.

$$C=\frac{1}{12n}\sum_{i=1}^{n}\left\{F\left(x_{(i)};\lambda,\theta\right)-\frac{2i-1}{2n}\right\}^2 \tag{17}$$

By substituting its cdf specified in Eq. (2), for $F\left(x_{(i)}\right)$in the previous equation, this function can be produced for the new mixture distribution (MEGD).

$$C=\frac{1}{12n}\sum_{i=1}^{n}\left\{\left(1+\frac{\lambda\theta x_{(i)}}{(\lambda+1)}\left(\frac{\theta x_{(i)}}{2}+1\right)\right)e^{-\theta x_{(i)}}-\frac{2i-1}{2n}\right\}^2 \tag{18}$$

We can determine the CVM estimates by maximizing in Eq. (18), by solving the following equations.

$$\frac{C(\lambda,\theta)}{\partial\theta}=\sum_{i=1}^{n}\left\{\left(1+\frac{\lambda\theta x_{(i)}}{(\lambda+1)}\left(\frac{\theta x_{(i)}}{2}+1\right)\right)e^{-\theta x_{(i)}}-\frac{2i-1}{2n}\right\}=0$$

$$\frac{C(\lambda,\theta)}{\partial\lambda}=\sum_{i=1}^{n}\left\{\left(1+\frac{\lambda\theta x_{(i)}}{(\lambda+1)}\left(\frac{\theta x_{(i)}}{2}+1\right)\right)e^{-\theta x_{(i)}}-\frac{2i-1}{2n}\right\}=0$$

Eq's. (11) and (12) provide $\zeta_i^{(1)}(\lambda,\theta)$ *and* $\zeta_i^{(2)}(\lambda,\theta)$, respectively.

15. METHOD FOR ANDERSON-DARLING

The method of Anderson-darling estimation was introduced by [1] in the context of statistical tests. By adapting to the new mixture distribution model (MEGD), the Anderson-darling estimates (ADEs) of λ *and* θ , the function is given by

$$A(\lambda,\theta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left(F\left(x_{(i)};\lambda,\theta\right)\right)+\log\left(1-F\left(x_{(i)};\lambda,\theta\right)\right)\right\} \quad (19)$$

Thus, the Anderson-Darling estimates can be obtained by solving the following equations simultaneously$\frac{A(\lambda,\theta)}{\partial\lambda}=0,\ \frac{A(\lambda,\theta)}{\partial\theta}=0$

$$\frac{A(\lambda,\theta)}{\partial\theta}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\frac{\zeta_i^{(1)}(\lambda,\theta)}{F\left(x_{(i)};\lambda,\theta\right)}-\frac{\zeta_i^{(1)}(\lambda,\theta)}{1-F\left(x_{(i)};\lambda,\theta\right)}\right\}=0$$

$$\frac{A(\lambda,\theta)}{\partial\lambda}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\frac{\zeta_i^{(2)}(\lambda,\theta)}{F\left(x_{(i)};\lambda,\theta\right)}-\frac{\zeta_i^{(2)}(\lambda,\theta)}{1-F\left(x_{(i)};\lambda,\theta\right)}\right\}=0$$

The value of $\zeta_i^{(1)}(\lambda,\theta)$ *and* $\zeta_i^{(2)}(\lambda,\theta)$ are provided in Eq's. (11) and (12), in that order.

16. ESTIMATION OF PARAMETERS

The new mixed distribution (MEGD) parameter's maximum likelihood estimates and Fisher's information matrix are provided in this section.

16.1 Maximum likelihood estimation (MLE) and fisher's information matrix

Consider $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from the new mixture distribution (MEGD) with parameter α,θ the likelihood function, which is defined as

$$L=(x;\lambda,\theta)=\prod_{i=1}^{n}f(x_i;\lambda,\theta)$$

$$L=\prod_{i=1}^{n}\frac{\theta}{\lambda+1}\left(\frac{1}{2}\lambda\theta^2x_i^2+1\right)e^{-\theta x_i}$$

Then, the log-likelihood function is given

$$\ell=\log L=n\log\theta-n\log(\lambda+1)+n\log\sum_{i=1}^{n}\left(\frac{1}{2}\lambda\theta^2x_i^2+1\right)-\theta\sum_{i=1}^{n}x_i$$

Differentiating with respect to θ *and* λ, we get

$$\frac{\partial\log L}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\left(\frac{(\lambda\theta x_i^2)}{\left(\frac{1}{2}\lambda\theta^2x_i^2+1\right)}\right)-\sum_{i=1}^{n}x_i=0 \quad (20)$$

$$\frac{\partial\log L}{\partial\lambda}=-\frac{n}{(\lambda+1)}+\sum_{i=1}^{n}\left(\frac{\left(\frac{1}{2}\theta^2x_i^2\right)}{\left(\frac{1}{2}\lambda\theta^2x_i^2+1\right)}\right)=0 \quad (21)$$

The maximum likelihood estimate of the parameters for the new mixture distribution is provided by Eq's. (20) and (21). The equation, however, cannot be solved analytically, so they used R programming and a data set to solve it numerically.

The asymptotic normality results are used to derive the confidence interval. Given that if $\hat{\lambda}=(\hat{\theta},\hat{\lambda})$ represents the MLE of $\lambda=(\theta,\lambda)$, the results can be expressed as follows:

$\sqrt{n}\left(\hat{\lambda}-\lambda\right)\rightarrow N_2(0,I^{-1}(\lambda))$

In this case, $I(\lambda)$represents Fisher's Information Matrix.

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial\theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial\theta\partial\lambda}\right) \\ E\left(\frac{\partial^2 \log L}{\partial\lambda\partial\theta}\right) & E\left(\frac{\partial^2 \log L}{\partial\lambda^2}\right) \end{pmatrix}$$

$$\left(\frac{\partial^2 \log L}{\partial\theta^2}\right) = \frac{n}{\theta^2} + \sum_{i=1}^{n}\left(\frac{\lambda x_i^2\left(\frac{1}{2}\lambda\theta^2 x_i^2 + 1 - \lambda\theta^2 x_i^2\right)}{\left(\frac{1}{2}\lambda\theta^2 x_i^2 + 1\right)^2}\right)$$

$$\left(\frac{\partial^2 \log L}{\partial\lambda^2}\right) = \frac{n}{(\lambda+1)^2} - \sum_{i=1}^{n}\left(\frac{\frac{1}{4}\theta^4 x_i^4}{\left(\frac{1}{2}\lambda\theta^2 + 1\right)^2}\right)$$

$$\left(\frac{\partial^2 \log L}{\partial\theta\partial\lambda}\right) = \sum_{i=1}^{n}\left(\frac{x_i^2\left(\frac{1}{2}\lambda^2\theta^4 x_i^4 + 1 - 2\lambda\theta^2 x_i^2 - \lambda\theta^2 x_i^2\right)}{\left(\frac{1}{2}\lambda\theta^2 x_i^2 + 1\right)^4}\right)$$

$$\left(\frac{\partial^2 \log L}{\partial\lambda\partial\theta}\right) = -\sum_{i=1}^{n}\left(\frac{x_i^2\theta^2\left(\frac{1}{4}\lambda^2\theta^4 x_i^4 + 1\right)}{\left(\frac{1}{2}\lambda\theta^2 x_i^2 + 1\right)^4}\right)$$

17. SIMULATION

This part uses a simulation study to evaluate the effectiveness and accuracy of the new mixture distribution's (MEGD) parameter estimation technique. The average bias (AB), average mean square error (AMSE), and parameter space $\xi = (\lambda\theta)$, are estimated.

$$AB = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\xi}_\iota - \xi\right) \qquad MSE_s = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\xi}_\iota - \xi\right)^2$$

Five different methods were used to estimate the parameters of the two-parameter new mixed distribution (MEGD). Moreover, the Kolmogorov-Smirnov (KS) test can be expressed as $= max\left|\hat{F}\left(x_{(i)}\right) - \left(\frac{i}{n+1}\right)\right|$. Table 1 lists the size of the samples (100) used in each scenario. Next, the average biassed (AB) and mean square error (MSE) were used to compare the parameters and approaches.

Table 1. A summary of the simulation's results.

n	Method	$\hat{\lambda}$	$\hat{\theta}$	$AB(\hat{\lambda})$	AMSE$(\hat{\lambda})$	$AB(\hat{\theta})$	$AMSE(\hat{\theta})$
100	LSE	9.999932	0.8010688	NA	0.2994652	-1.893048	0.2994652
	WLS	5.781467	1.05995	NA	153.4519	6.814661	153.4519
	MPS	1.5	2.0	0.25619	NA	3.628930	10.62661
	CVM	13987930	1.837299	-	-	2.807808	10.41995
	AD	6.546030	1.225795	2.808634	92151944	2.808634	10.15757
	KS	0.1099256	0.6001768				

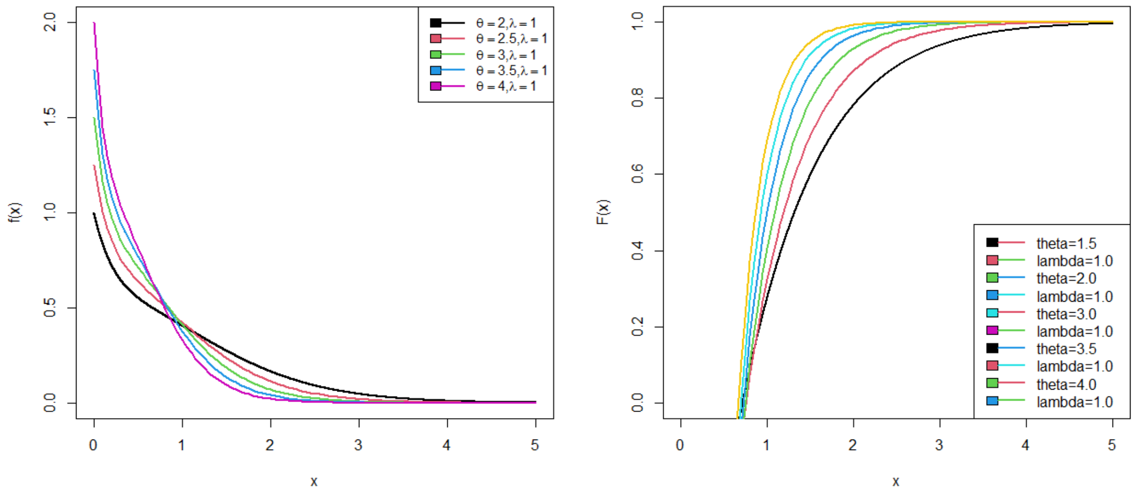

Figure 1. Pdf and Cdf plot of the New distribution (MEGD)

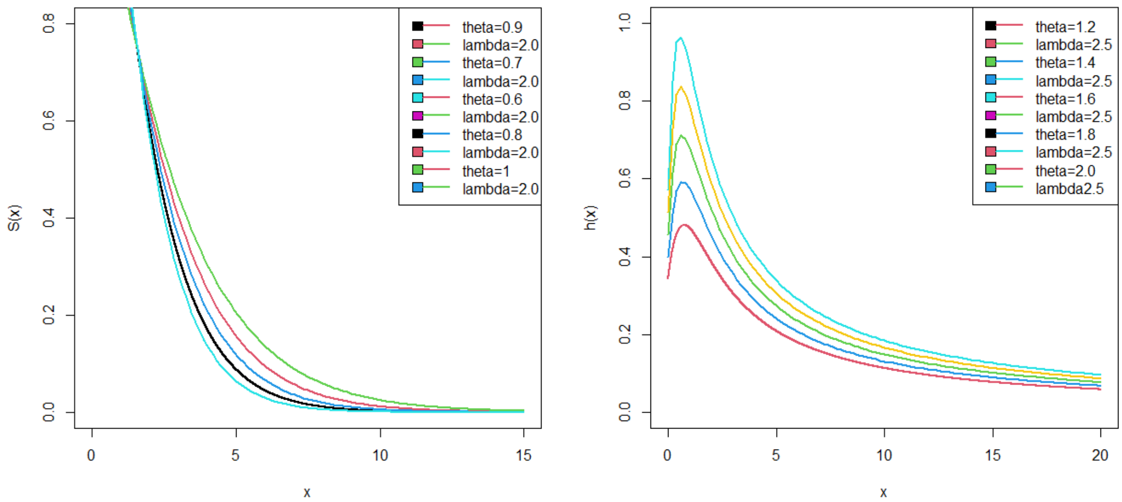

Figure 2.Survival and Hazard function plot of the New distribution (MEGD)

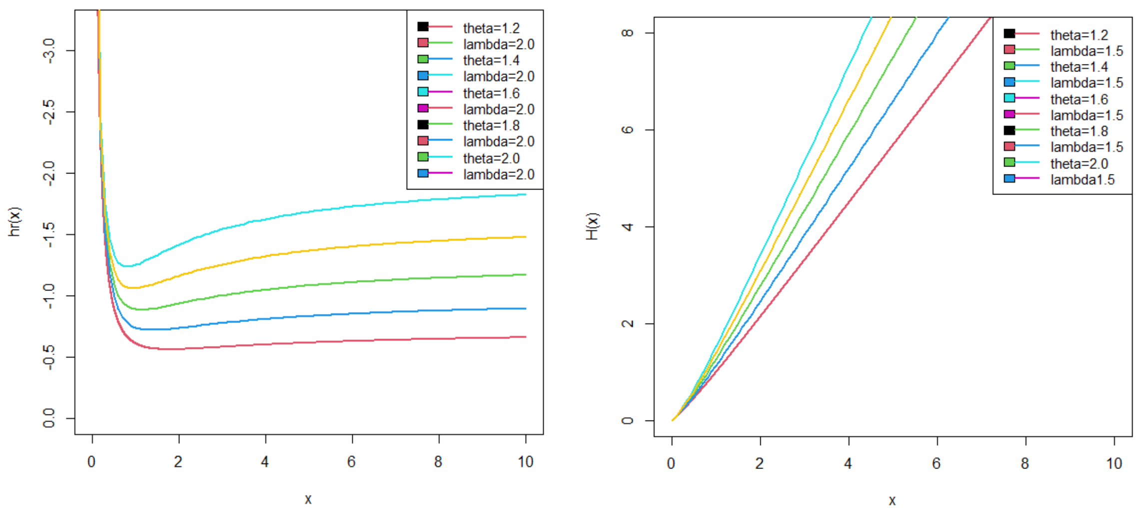

Figure 3. Revers hazard and Cumulative hazard function plot of the New distribution (MEGD)

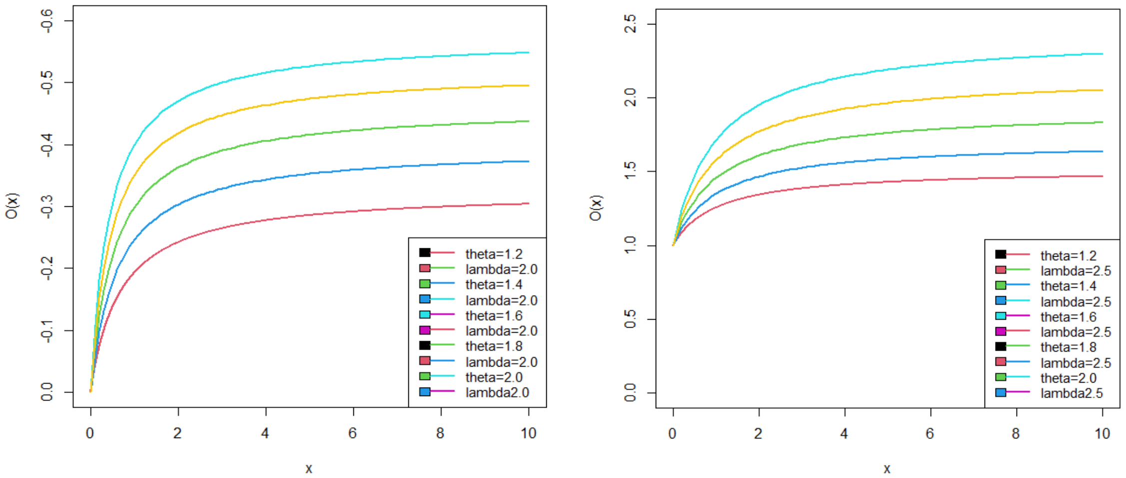

Figure 4. Odds rate and Quantial function plot of the New distribution (MEGD)

18. APPLICATIONS

Dat set 1: This data includes the life expectancy (in years) of forty patients with leukemia, a blood malignancy, from one of Saudi Arabia's Ministry of Health facilities, as published in (25). This real information is

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348
3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244	4.323
4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	5.381	

Data set 2: The data under consideration are the life times of 20 leukemia patients who were treated by a certain drug (20). The data are

1.013	1.034	1.109	1.169	1.226	1.509	1.533	1.563	1.716	1.929	1.965	2.061	2.344	2.546
2.626	2.778	2.951	3.413	4.118	5.136								

The Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Akaike Information Criteria Corrected (AICC), and $-2\log L$.are used to compare the goodness of fit of the fitted distribution.
The following formula can be used to determine AIC, BIC, AICC, and $-2\log L$..

$$AIC = 2k - 2\log L\,, \qquad BIC = k\log n - 2\log L \ \ and\ AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$

Where, $k =$ number of parameters, n sample size and $-2\log L$ is the maximized value of loglikelihood function.

Table 2. MLEs of the fitted distribution for the provided data set 1 are AIC, BIC, AICC, and $-2\log L$.

Distribution	ML Estimates	SE	-2log L	AIC	BIC	AICC
New mixture of exponential and gamma distribution	$\hat{\lambda} = 1.75028$ $\hat{\theta} = 0.99031$	0.87027 0.12148	144.0257	148.0257	151.3528	148.3590
Lindely	$\hat{\theta} = 0.25770$	0.061617	156.5028	158.5028	160.1664	158.6080
Shanker	$\hat{\theta} = 0.549721$	0.058062	144.7945	155.9545	157.6181	156.0597

Rama	$\hat{\theta} = 1.101465$	0.080551	143.3158	154.3158	147.1023	154.4210
Exponential	$\hat{\theta} = 0.318938$	0.051070	167.1353	169.1353	170.7988	169.0405
Aradhana	$\hat{\theta} = 0.750601$	0.071081	149.4283	151.4283	153.0918	151.5335
Akash	$\hat{\theta} = 0.801683$	0.071209	149.0561	151.0561	152.7196	151.1613
Ishita	$\hat{\theta} = 0.80668$	0.06521	147.9967	149.9967	151.6603	150.1019
Quasi Lindely	$\hat{\theta} = 0.63762$ $\hat{\alpha} = 0.0010$	0.145085 0.502755	149.1123	153.1123	156.4394	153.4366
Quasi Shanker	$\hat{\theta} = 0.93565$ $\hat{\alpha} = 21.02154$	0.09713 57.71126	147.4368	151.4368	154.8146	151.7611
Quasi Aradhana	$\hat{\theta} = 0.63762$ $\hat{\alpha} = 0.00100$	0.08172 0.15278	203.1178	207.1178	210.5049	207.4421
Quasi Sujatha	$\hat{\theta} = 0.86070$ $\hat{\alpha} = 0.00100$	0.09772 0.40129	145.2141	149.2141	152.5412	149.5384

Table 3. MLEs of the fitted distribution for the provided data set 2 are AIC, BIC, AICC, and $-2\log L$.

Distribution	ML Estimates	SE	-2log L	AIC	BIC	AICC
New mixture of exponential and gamma distribution	$\hat{\lambda} = 5.85527$ $\hat{\theta} = 1.29745$	2.89211 0.51146	53.59311	57.59311	59.48199	58.34311
Lindely	$\hat{\theta} = 0.70768$	0.12007	64.02158	66.02158	66.96602	66.2438
Shanker	$\hat{\theta} = 0.71243$	0.10777	63.08856	65.08856	66.033	65.3107
Rama	$\hat{\theta} = 1.37842$	0.14153	62.41991	64.41991	65.36435	64.6421
Exponential	$\hat{\theta} = 0.44632$	0.10239	68.65501	70.65501	71.59945	70.8772
Aradhana	$\hat{\theta} = 0.9855$	0.13594	60.60053	62.60053	63.54497	62.8227
Akash	$\hat{\theta} = 0.02970$	0.13179	62.69158	64.69158	65.63602	64.9138
Ishita	$\hat{\theta} = 0.99759$	0.11340	62.74297	64.74297	65.68741	64.9651
Quasi Lindely	$\hat{\theta} = 0.89223$ $\hat{\alpha} = 0.00100$	0.12540 (NaN)	57.9066	61.9066	63.79554	62.6566
Quasi Aradhana	$\hat{\theta} = 0.89237$ $\hat{\alpha} = 0.00100$	0.19067 0.45574	84.24628	88.2462	90.1351	88.9962
Quasi Sujatha	$\hat{\theta} = 1.17372$ $\hat{\alpha} = 0.00100$	0.12441 (NaN)	55.9815	59.9815	61.8703	60.7315

19. RESULTS AND DISCUSSION

Table 1 shows different methods to estimate the parameters of the new distribution as a mixture of exponential and gamma of the simulation's results. Tables 2 and 3 show that the MEGD distribution has smaller AIC, BIC, AICC, and values than the new distribution, which is a mixture of exponential and gamma distribution, Lindely, Shanker, Rama, Exponential, Aradhana, Akash, Ishita, distribution, and Lindely, Aradhana, Sujatha, Shanker Quasi distribution. This implies that the data is better fitted by the new mixture distribution (MEGD). As a result, the combination of exponential and gamma distributions offers a better fit than the other distributions.

20. CONCLUSION

The new mixture of exponential and gamma distribution (MEGD) is presented in this article as a model for lifetime data. The article contains several exceptional cases that it possesses. There have been derived several statistical properties of the suggested distribution. Including reliability analysis, moments, the moment-generating function, Bonferroni and Lorenz curves, entropy, as well as order statistics. Using the maximum likelihood estimation method, the unknown parameters for a new mixture distribution were inferred. The estimates have been assessed in several simulated investigations. Utilizing the maximum likelihood method to estimate the parameters results in a satisfactory performance. By using several goodness-of-fit criteria, the results show that the new mixture of exponential and gamma distributions (MEGD) performs better than other well-known distributions.

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